

THE LOEWNER-NIRENBERG PROBLEM IN SINGULAR DOMAINS

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ABSTRACT. We study the asymptotic behaviors of solutions of the Loewner-Nirenberg problem in singular domains and prove that the solutions are well approximated by the corresponding solutions in tangent cones at singular points on the boundary. The conformal structure of the underlying equation plays an essential role in the derivation of the optimal estimates.

1. INTRODUCTION

Assume $\Omega \subset \mathbb{R}^n$ is a domain, for $n \geq 3$. We consider the following problem:

$$(1.1) \quad \Delta u = \frac{1}{4}n(n-2)u^{\frac{n+2}{n-2}} \quad \text{in } \Omega,$$

$$(1.2) \quad u = \infty \quad \text{on } \partial\Omega.$$

This is the so-called Loewner-Nirenberg problem, also known as the singular Yamabe problem. For a large class of domains Ω , (1.1) and (1.2) admit a unique positive solution $u \in C^\infty(\Omega)$. Geometrically, $u^{\frac{4}{n-2}} \sum_{i=1}^n dx_i \otimes dx_i$ is a complete metric with the constant scalar curvature $-n(n-1)$ on Ω .

The two dimensional counterpart is given by, for $\Omega \subset \mathbb{R}^2$,

$$(1.3) \quad \Delta u = e^{2u} \quad \text{in } \Omega.$$

More generally, we can study, for a function f ,

$$\Delta u = f(u) \quad \text{in } \Omega.$$

For bounded domains Ω , let d be the distance function to $\partial\Omega$. If $\partial\Omega$ is C^2 , then d is a C^2 -function near $\partial\Omega$. In a pioneering work, Loewner and Nirenberg [17] studied asymptotic behaviors of solutions of (1.1) and (1.2) and proved, for d sufficiently small,

$$(1.4) \quad |d^{\frac{n-2}{2}}u - 1| \leq Cd,$$

where C is a positive constant depending only on certain geometric quantities of $\partial\Omega$. This follows from a comparison of u and the corresponding solutions in the interior and exterior tangent balls. This result has been generalized to more general f and up to higher order terms, for example, by Brandle and Marcus [3], Diaz and Letelier [6], and Kichenassamy [12]. All these results require $\partial\Omega$ to have some degree of regularity. The case where $\partial\Omega$ is singular was studied by del Pino and Letelier [5], and Marcus

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and Veron [18]. However, there are no explicit estimates in neighborhoods of singular boundary points in these works.

In [9], we studied the asymptotic behaviors of solutions of (1.3) and (1.2) in planar singular domains, and proved that these solutions are well approximated by the corresponding solutions in tangent cones near isolated singular points on the boundary. Based on a combination of conformal transforms and the maximum principle, we derived an optimal estimate.

In this paper, we study the asymptotic behaviors of solutions of (1.1) and (1.2) near singular points on $\partial\Omega$. Similarly as in [9], we prove that the solutions of (1.1) and (1.2) can be approximated by the corresponding solutions in tangent cones at singular points on the boundary.

Presumably, it is more difficult to discuss solutions of (1.1) and (1.2) for $n \geq 3$ than those of (1.3) and (1.2) for $n = 2$, for several reasons. First, the conformal invariance of domains is more restrictive for $n \geq 3$. For example, cones are not conformal to each other unless they are conjugate. Second, there are no explicit solutions of (1.1) and (1.2) in cones in general. Third, the type of the boundary singularity is more diverse. We need to introduce new techniques to address these issues.

Our main result in this paper is given by the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $x_0 \in \partial\Omega$ and, for some integer $k \leq n$, let $\partial\Omega$ in a neighborhood of x_0 consist of k $C^{1,1}$ -hypersurfaces S_1, \dots, S_k intersecting at x_0 with the property that the normal vectors of S_1, \dots, S_k at x_0 are linearly independent. Suppose $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2), and $u_{V_{x_0}}$ is the corresponding solution in the tangent cone V_{x_0} of Ω at x_0 . Then, there exist a constant r and a $C^{1,1}$ -diffeomorphism $T: B_r(x_0) \rightarrow T(B_r(x_0)) \subseteq \mathbb{R}^n$, with $T(\Omega \cap B_r(x_0)) = V_{x_0} \cap T(B_r(x_0))$ and $T(\partial\Omega \cap B_r(x_0)) = \partial V_{x_0} \cap T(B_r(x_0))$, such that, for any $x \in B_{r/2}(x_0)$,*

$$(1.5) \quad \left| \frac{u(x)}{u_{V_{x_0}}(Tx)} - 1 \right| \leq C|x - x_0|,$$

where C is a positive constant depending only on n and the geometry of $\partial\Omega$.

The estimate (1.5) generalizes (1.4) to singular domains and is optimal. The power one of the distance in the right-hand side cannot be improved without better regularity assumptions of the boundary. This estimate resembles a similar estimate for the equation for the positive scalar curvature near isolated singular points, established in [4]. A refined version was proved in [13]. Refer to [10] for more general equations.

In the proof of Theorem 1.1, we will construct the map T , which is determined by the distances to S_i . The concept of tangent cones will be introduced in Section 4.

We now describe briefly the proof of Theorem 1.1, which is based on a combination of affine transforms, conformal transforms and the maximum principle. To make a comparison, we note that, if the domain Ω is $C^{1,1}$, we can place an interior tangent ball and an exterior tangent ball at each of the boundary point and then compare the solution in Ω with the solution in the interior tangent ball and with the solution outside the exterior tangent ball. Refer to the proof of Theorem 3.1 for details. Now we assume

that $\partial\Omega$ near a boundary point x_0 consists of k $C^{1,1}$ -hypersurfaces S_1, \dots, S_k intersecting at x_0 , for some $k \leq n$. Then, the tangent planes P_1, \dots, P_k of S_1, \dots, S_k at x_0 naturally bound a cone V_{x_0} , which is the tangent cone of Ω at x_0 . Our goal is to compare the solution u in Ω near x_0 with the solution $u_{V_{x_0}}$ in V_{x_0} . We note that a given point x in Ω may not necessarily be a point in the tangent cone V_{x_0} . So as a part of the comparison of u with $u_{V_{x_0}}$, we need to construct a map T , which maps Ω near x_0 onto V_{x_0} near x_0 , and to compare $u(x)$ with $u_{V_{x_0}}(Tx)$. We achieve this in two steps.

In the first step, we construct two sets \tilde{B} and \hat{B} with the property $\tilde{B} \subseteq \Omega \subseteq \hat{B}$, where \tilde{B} serves the same role as the interior tangent ball in the $C^{1,1}$ -case and \hat{B} serves the same role as the complement of the exterior tangent ball in the $C^{1,1}$ -case. To construct such sets \tilde{B} and \hat{B} , we first place two balls tangent to P_i at p_i , the closest point to x on S_i , for each $i = 1, \dots, k$. We can assign an orientation such that we can identify one of these balls as *interior* and another as *exterior*. As a result, we have k interior balls and k exterior balls. Based on how Ω is formed by S_1, \dots, S_k near x_0 , we can form \tilde{B} from the interior balls and \hat{B} from the complement of the exterior balls. By such a construction, \tilde{B} and \hat{B} are conformal to infinite cones \tilde{V} and \hat{V} .

In the second step, we compare the solution u in Ω near x_0 with the solution $u_{V_{x_0}}$ in the tangent cone V_{x_0} . To this end, we first compare u with the solutions \tilde{u} and \hat{u} in \tilde{B} and \hat{B} , respectively, and then compare \tilde{u} and \hat{u} with $u_{V_{x_0}}$. The comparison of u with \tilde{u} and \hat{u} is based on a simple application of the maximum principle. However, the comparison of \tilde{u} and \hat{u} with $u_{V_{x_0}}$ is delicate and occupies a large portion of the paper. Since \tilde{B} and \hat{B} are conformal to infinite cones \tilde{V} and \hat{V} , solutions \tilde{u} and \hat{u} are related to solutions \tilde{v} and \hat{v} in \tilde{V} and \hat{V} by conformal factors. As a result, we need to compare $u_{V_{x_0}}$ with \tilde{v} and \hat{v} , all of which are solutions in cones. Such a comparison is based on an anisotropic gradient estimate for solutions in cones. The map T from Ω to V_{x_0} near x_0 mentioned above is constructed through the signed distances from x in Ω to S_1, \dots, S_k and from the corresponding point in V_{x_0} to the faces of V_{x_0} .

The conformal structure of the equation (1.1) plays an essential role. Without such a structure, we will be unable to derive the optimal estimate (1.5). Specifically, if the power $\frac{n+2}{n-2}$ in (1.1) is replaced by other constant $p > 1$, we can bound the left-hand side of (1.5) by $C|x - x_0|^\alpha$, for some constant $\alpha \in (0, 1]$, depending on n and p . For general p , there is no conformal structure to utilize. The resulted α in fact is much smaller than 1. We will not pursue this issue in the present paper.

The paper is organized as follows. In Section 2, we discuss the existence of solutions of (1.1)-(1.2) in bounded Lipschitz domains and in a certain class of infinite cones. In Section 3, we prove some basic estimates for asymptotic behaviors near boundary and in particular an anisotropic gradient estimate for solutions in cones. In Section 4, we introduce the class of domains to be discussed in this paper and analyze their tangent cones. In Section 5, we compare solutions in different cones. In Section 6, we study the asymptotic expansions near singular points bounded by $C^{1,1}$ -hypersurfaces and prove Theorem 1.1. In Section 7, we discuss the asymptotic expansions in more general domains.

2. EXISTENCE OF SOLUTIONS

In this section, we discuss the existence of solutions of (1.1)-(1.2) in several classes of domains.

First, we introduce some notations. Let $x_0 \in \mathbb{R}^n$ be a point and $r > 0$ be a constant. Set, for any $x \in B_r(x_0)$,

$$(2.1) \quad u_{r,x_0}(x) = \left(\frac{2r}{r^2 - |x - x_0|^2} \right)^{\frac{n-2}{2}}.$$

Then, u_{r,x_0} is a solution of (1.1)-(1.2) in $\Omega = B_r(x_0)$. With $d(x) = r - |x - x_0|$, the distance to the sphere $\partial B_r(x_0)$ from $x \in B_r(x_0)$, we have

$$u_{r,x_0} = d^{-\frac{n-2}{2}} \left(1 - \frac{d}{2r} \right)^{-\frac{n-2}{2}}.$$

Set, for any $x \in \mathbb{R}^n \setminus B_r(x_0)$,

$$(2.2) \quad v_{r,x_0}(x) = \left(\frac{2r}{|x - x_0|^2 - r^2} \right)^{\frac{n-2}{2}}.$$

Then, v_{r,x_0} is a solution of (1.1)-(1.2) in $\Omega = \mathbb{R}^n \setminus B_r(x_0)$. With $d(x) = |x - x_0| - r$, the distance to the sphere $\partial B_r(x_0)$ from $x \in \mathbb{R}^n \setminus B_r(x_0)$, we have

$$v_{r,x_0} = d^{-\frac{n-2}{2}} \left(1 + \frac{d}{2r} \right)^{-\frac{n-2}{2}}.$$

These two solutions play an important role in this paper.

Now, we quote a well-known result.

Theorem 2.1. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n . Then, (1.1) and (1.2) admit a unique positive solution $u \in C^\infty(\Omega)$.*

Loewner and Nirenberg [17] proved the uniqueness for C^2 -domains. In fact, the uniqueness holds for Lipschitz domains as stated in Theorem 2.1.

Next, we state a basic result which will be needed later.

Lemma 2.2. *Let Ω be a domain in \mathbb{R}^n , and u and v be two nonnegative solutions of (1.1). Then, $u + v$ is a nonnegative supersolution of (1.1).*

We omit the proof as it is based on a straightforward calculation.

In the following, we discuss (1.1)-(1.2) in infinite cones. Throughout this paper, cones are always solid.

Theorem 2.3. *For a fixed integer $2 \leq k \leq n$, let V_k be an infinite cone in \mathbb{R}^k such that $V_k \cap \mathbb{S}^{k-1}$ is a Lipschitz domain in \mathbb{S}^{k-1} . Then, there exists a solution $u \in C^\infty(V)$ of (1.1)-(1.2) in $V = V_k \times \mathbb{R}^{n-k}$ and u is a function in x_1, \dots, x_k and satisfies, for any $\lambda > 0$ and any $x \in V$,*

$$(2.3) \quad u(\lambda x) = \lambda^{-\frac{n-2}{2}} u(x).$$

The scaling property (2.3) will be used repeatedly in the rest of the paper.

Proof. Let (r, θ) be the polar coordinates in \mathbb{R}^k . Then,

$$\Delta_{\mathbb{R}^k} = \frac{\partial^2}{\partial r^2} + \frac{k-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{k-1}},$$

where $\Delta_{S^{k-1}}$ is the Laplace-Beltrami operator on the unit sphere S^{k-1} . Set

$$u(x) = r^\alpha g(\theta),$$

and substitute such a u in (1.1) and (1.2). It is easy to see $\alpha = -\frac{n-2}{2}$ and (1.1)-(1.2) hold if

$$(2.4) \quad \Delta_{S^{k-1}} g + \left(-\frac{n-2}{2}\right) \left(k-1-\frac{n}{2}\right) g = \frac{1}{4} n(n-2) g^{\frac{n+2}{n-2}} \quad \text{in } S_k,$$

$$(2.5) \quad g = \infty \quad \text{on } \partial S_k,$$

where $S_k = V_k \cap S^{k-1}$. We will prove that there exists a nonnegative solution g of (2.4)-(2.5).

For each integer $i \geq 1$, there exists a solution $g_{(i)} \in C(\overline{S_k}) \cap C^\infty(S_k)$ of

$$\begin{aligned} \Delta_{S^{k-1}} g_{(i)} + \left(-\frac{n-2}{2}\right) \left(k-1-\frac{n}{2}\right) g_{(i)} &= \frac{1}{4} n(n-2) g_{(i)}^{\frac{n+2}{n-2}} \quad \text{in } S_k, \\ g_{(i)} &= i \quad \text{on } \partial S_k. \end{aligned}$$

The proof is based on a standard iteration. We now claim $\Delta_{S^{k-1}} g_{(i)} \geq 0$ in S_k , i.e.,

$$(2.6) \quad \frac{n-2}{2} \left(k-1-\frac{n}{2}\right) g_{(i)} + \frac{1}{4} n(n-2) g_{(i)}^{\frac{n+2}{n-2}} \geq 0.$$

First, (2.6) obviously holds if $k-1 \geq n/2$. Next, we consider the case $k-1 < n/2$. If (2.6) is violated somewhere, then g must assume its minimum at some point θ_0 in the set

$$\left\{ \theta \in S_k^n : \frac{n-2}{2} \left(k-1-\frac{n}{2}\right) g_{(i)} + \frac{n(n-2)}{4} g_{(i)}^{\frac{n+2}{n-2}} < 0 \right\}.$$

On the other hand, we have $\Delta_{S^{k-1}} g_{(i)}(\theta_0) \geq 0$, which leads to a contradiction.

By taking a difference, we have

$$\Delta_{S^{k-1}}(g_{(i)} - g_{(i-1)}) = c_i(g_{(i)} - g_{(i-1)}) \quad \text{in } S_k,$$

where c_i is a nonnegative function in S_k . We need (2.6) to prove $c_i \geq 0$. The maximum principle implies, $g_{(i)} \geq g_{(i-1)}$ for any $i \geq 2$.

For any $\theta \in S_k = V_k \cap S^{k-1}$, write $x_\theta = (\theta, 0_{\mathbb{R}^{n-k}}) \in \mathbb{R}^n$. For an arbitrarily fixed $\theta_0 \in S_k$, take a ball $B_{r_0}(x_{\theta_0}) \subset V = V_k \times \mathbb{R}^{n-k}$. Then, $u_{(i)}(x) = (\sqrt{x_1^2 + \dots + x_k^2})^{-\frac{n-2}{2}} g_{(i)}(\theta)$ satisfies

$$\begin{aligned} \Delta u_{(i)} &= \frac{1}{4} n(n-2) u_{(i)}^{\frac{n+2}{n-2}} \quad \text{in } V, \\ u_{(i)} &= (\sqrt{x_1^2 + \dots + x_k^2})^{-\frac{n-2}{2}} i \quad \text{on } \partial V. \end{aligned}$$

Let $u_{r_0, x_{\theta_0}}$ be the solution of (1.1)-(1.2) in $B_{r_0}(x_{\theta_0})$, given by (2.1). By the maximum principle, we have, for any $x \in B_{r_0}(x_{\theta_0})$,

$$u_{(i)}(x) \leq u_{r_0, x_{\theta_0}}(x).$$

Then, for any $\theta \in S_k$ with $x_\theta \in B_{r_0}(x_0)$,

$$g_{(i)}(\theta) = u_{(i)}(x_\theta) \leq u_{r_0, x_{\theta_0}}(x_\theta).$$

Therefore, there exists a $g \in C^\infty(S_k)$ such that $g_{(i)} \rightarrow g$ in $C_{loc}^m(S_k)$, for any positive integer m . Since $g_{(i)}$ equals i on ∂S_k , g is a solution of (2.4)-(2.5). \square

If the cone V as in Lemma 2.3 is Lipschitz near the origin, then the nonnegative solution u of (1.1)-(1.2) for V is unique. To verify this, let \tilde{u} be another nonnegative solution of (1.1)-(1.2). By Lemma 2.2 and the maximum principle, we have, for any ϵ small and $r \gg |x|$ large,

$$\begin{aligned} \tilde{u}(x) &\leq u(x - \epsilon e) + u_{r,0}(x), \\ \tilde{u}(x) &\geq u(x + \epsilon e) - u_{r,0}(x), \end{aligned}$$

where e is some unit vector in V and $u_{0,r}$ is the solution of (1.1)-(1.2) in B_r . Letting $\epsilon \rightarrow 0$ and $r \rightarrow \infty$, we obtain $\tilde{u}(x) = u(x)$, which implies the uniqueness.

3. BASIC ESTIMATES

In this section, we prove several basic estimates concerning asymptotic behaviors of solutions near boundary. First, we study the asymptotic behavior near $C^{1,\alpha}$ -portions of $\partial\Omega$.

Theorem 3.1. *Let Ω be a bounded domain in \mathbb{R}^n and $\partial\Omega$ be $C^{1,\alpha}$ near $x_0 \in \partial\Omega$ for some $\alpha \in (0, 1]$. Suppose $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2). Then,*

$$|d^{\frac{n-2}{2}}u - 1| \leq Cd^\alpha \quad \text{in } \Omega \cap B_r(x_0),$$

where d is the distance to $\partial\Omega$, and r and C are positive constants depending only on n , α and the geometry of Ω .

Proof. We take $R > 0$ sufficiently small such that $\partial\Omega \cap B_R(x_0)$ is $C^{1,\alpha}$. We fix an $x \in \Omega \cap B_{R/4}(x_0)$ and take $p \in \partial\Omega$, also near x_0 , such that $d(x) = |x - p|$. Then, $p \in \partial\Omega \cap B_{R/2}(x_0)$. By a translation and rotation, we assume $p = 0$ and the x_n -axis is the interior normal to $\partial\Omega$ at 0. Then, x is on the positive x_n -axis, with $d = d(x) = |x|$, and $P = \{x_n = 0\}$ is the tangent plane of $\partial\Omega$ at 0. Moreover, a portion of $\partial\Omega$ near 0 can be expressed by a $C^{1,\alpha}$ -function φ in $B'_R \subset \mathbb{R}^{n-1}$, with $\varphi(0) = 0$ and

$$(3.1) \quad |\varphi(x')| \leq M|x'|^{1+\alpha} \quad \text{for any } x' \in B'_R.$$

Here, M is a positive constant chosen to be uniform, independent of x .

We first consider the case $\alpha = 1$. For any $r > 0$, the lower semi-sphere of

$$x_1^2 + \dots + x_{n-1}^2 + (x_n - r)^2 = r^2$$

satisfies $x_n \geq |x'|^2/(2r)$. By fixing a constant r sufficiently small, (3.1) implies

$$B_r(re_n) \subset \Omega \text{ and } B_r(-re_n) \cap \Omega = \emptyset.$$

Let u_{r,re_n} and $v_{r,-re_n}$ be the solutions of (1.1)-(1.2) in $B_r(re_n)$ and $\mathbb{R}^n \setminus B_r(-re_n)$, given by (2.1) and (2.2), respectively. Then, by the maximum principle, we have

$$v_{r,-re_n} \leq u \leq u_{r,re_n} \quad \text{in } B_r(re_n).$$

For the x above in the positive x_n -axis with $|x| = d < r$, we obtain

$$d^{-\frac{n-2}{2}} \left(1 + \frac{d}{2r}\right)^{-\frac{n-2}{2}} \leq u \leq d^{-\frac{n-2}{2}} \left(1 - \frac{d}{2r}\right)^{-\frac{n-2}{2}}.$$

This implies the desired result for $\alpha = 1$.

Next, we consider $\alpha \in (0, 1)$. Recall that x is in the positive x_n -axis and $|x| = d$. We first note

$$(3.2) \quad |x'|^{1+\alpha} \leq d^{1+\alpha} + \frac{1}{d^{1-\alpha}} |x'|^2 \quad \text{for any } x' \in \mathbb{R}^{n-1}.$$

This follows from the Hölder inequality, or more easily, by considering $|x'| \leq d$ and $|x'| \geq d$ separately. Let $r = d^{1-\alpha}/(2M)$ and q be the point on the positive x_n -axis such that $|q| = Md^{1+\alpha} + r$. By taking d sufficiently small, (3.1) and (3.2) imply

$$B_r(q) \subset \Omega \text{ and } B_r(-q) \cap \Omega = \emptyset.$$

Let $u_{r,q}$ and $v_{r,-q}$ be the solutions of (1.1)-(1.2) in $B_r(q)$ and $\mathbb{R}^n \setminus B_r(-q)$, given by (2.1) and (2.2), respectively. Then, by the maximum principle, we have

$$v_{r,-q} \leq u \leq u_{r,q} \quad \text{in } B_r(q).$$

For the x above, $\text{dist}(x, \partial B_r(q)) = d - Md^{1+\alpha}$ and $\text{dist}(x, \partial B_r(-q)) = d + Md^{1+\alpha}$. Evaluating at such an x , we obtain

$$\begin{aligned} & (d + Md^{1+\alpha})^{-\frac{n-2}{2}} \left(1 + \frac{M}{d^{1-\alpha}}(d + Md^{1+\alpha})\right)^{-\frac{n-2}{2}} \\ & \leq u \leq (d - Md^{1+\alpha})^{-\frac{n-2}{2}} \left(1 + \frac{M}{d^{1-\alpha}}(d - Md^{1+\alpha})\right)^{-\frac{n-2}{2}}. \end{aligned}$$

This implies the desired result for $\alpha \in (0, 1)$. \square

If $\partial\Omega$ is $C^{1,\alpha}$ near x_0 for some $\alpha \in (0, 1]$, then the tangent cone V_{x_0} of Ω at x_0 is a half space and $v(x) = d(x)^{-\frac{n-2}{2}}$ is the solution of Loewner-Nirenberg problem in V_{x_0} , where d is the distance to ∂V_{x_0} .

Next, we prove a preliminary result for domains with singularity. We note that a finite circular cone is determined by its vertex, its axis, its height and its opening angle. The height and the opening angle are often referred to as the size of the cone. Here, cones are solid. We point out that we do not assume the boundedness of the domains in the next result.

Lemma 3.2. *Let Ω be a domain satisfying the uniform exterior cone condition in \mathbb{R}^n . Suppose $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2). Then there exists a constant $\delta > 0$, depending only on the size of exterior cones, such that, for any $x \in \Omega$ with $d(x) < \delta$,*

$$C^{-1} \leq d(x)^{\frac{n-2}{2}} u(x) \leq 2^{\frac{n-2}{2}},$$

where $d(x)$ is the distance from x to $\partial\Omega$, and C is a constant depending only on n and the size of exterior cones.

Proof. We take an arbitrarily fixed $x \in \Omega$. Then, $B_{d(x)}(x) \subseteq \Omega$. Let $u_{d(x),x}$ be the solution of (1.1)-(1.2) in $B_{d(x)}(x)$, given by (2.1). By the maximum principle, we have

$$u(x) \leq u_{d(x),x}(x) = d(x)^{-\frac{n-2}{2}} \left(1 - \frac{d(x)}{2d(x)}\right)^{-\frac{n-2}{2}} = 2^{\frac{n-2}{2}} d(x)^{-\frac{n-2}{2}}.$$

Next, we assume $d(x) = |x - p|$, for some $p \in \partial\Omega$. There exists a finite circular cone V_p , with the vertex p , the axis e_p , the height h , and the opening angle 2θ , such that $V_p \subseteq \Omega^c$. Here, we can assume h and θ are constants independent of the choice of $p \in \partial\Omega$. We further assume

$$(3.3) \quad d(x) < h(1 + \frac{1}{\sin \theta})^{-1}.$$

Set $\tilde{p} = p + \frac{1}{\sin \theta} d(x) e_p$. Then, $B_{d(x)}(\tilde{p}) \subseteq \Omega^c$. Let $v_{d(x),\tilde{p}}$ be the solution of (1.1)-(1.2) in $\mathbb{R}^n \setminus B_{d(x)}(\tilde{p})$, given by (2.2). By the maximum principle, we have

$$\begin{aligned} u(x) &\geq v_{d(x),\tilde{p}}(x) \geq \left(d(x) + \frac{1}{\sin \theta} d(x)\right)^{-\frac{n-2}{2}} \left(1 + \frac{d(x) + \frac{1}{\sin \theta} d(x)}{2d(x)}\right)^{-\frac{n-2}{2}} \\ &\geq C d(x)^{-\frac{n-2}{2}}. \end{aligned}$$

We conclude the desired estimate. \square

Lemma 3.2 demonstrates that, for any $x \in \Omega$, the values of solutions in $B_{d(x)/2}(x)$ are comparable.

Remark 3.3. The requirement on $d(x) < \delta$ in Lemma 3.2 is due to (3.3), where h is the height of the exterior cone. If at each point $p \in \partial\Omega$, there exists an infinite exterior cone with a fixed angle θ , then Lemma 3.2 holds for all $x \in \Omega$. A similar remark also holds for Lemma 3.4 below.

We next derive estimates of derivatives.

Lemma 3.4. *Let Ω be a domain satisfying the uniform exterior cone condition in \mathbb{R}^n . Suppose $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2). Then there exists a constant $\delta > 0$, depending only on the size of exterior cones, such that, for any $x \in \Omega$ with $d(x) < \delta$,*

$$(3.4) \quad d(x)|Du(x)| + d^2(x)|D^2u(x)| \leq Cu(x),$$

where $d(x)$ is the distance from x to $\partial\Omega$, and C is a constant depending only on n and the size of exterior cones.

Proof. We take an arbitrary $\beta \in (0, 1)$. By the standard interior estimates, we have, for any $x \in \Omega$ with $d = \text{dist}(x, \partial\Omega)$,

$$\begin{aligned} & d^\beta [u]_{C^\beta(B_{d/8}(x))} + d|u|_{L^\infty(B_{d/8}(x))} + d^{1+\beta} [Du]_{C^\beta(B_{d/8}(x))} \\ & \leq C\{|u|_{L^\infty(B_{d/4}(x))} + d^2 |u|^{\frac{n+2}{n-2}}|_{L^\infty(B_{d/4}(x))}\}, \end{aligned}$$

and

$$\begin{aligned} & d^2 |D^2 u|_{L^\infty(B_{d/16}(x))} + d^{2+\beta} [D^2 u]_{C^\beta(B_{d/8}(x))} \\ & \leq C\{|u|_{L^\infty(B_{d/8}(x))} + d^2 |u|^{\frac{n+2}{n-2}}|_{L^\infty(B_{d/8}(x))} + d^{2+\beta} [u]^{\frac{n+2}{n-2}}|_{C^\beta(B_{d/8}(x))}\}, \end{aligned}$$

where C is a positive constant depending only on n and β . Then, Lemma 3.2 implies the desired result. \square

In (3.4), the gradients at points are estimated in terms of their distances to the boundary. This estimate is not sufficient in many applications. In the next result, we estimate the directional derivatives in cones along certain directions in terms of the distance to the corresponding faces forming the boundary of the cones. Such an anisotropic gradient estimate plays a fundamental role in this paper.

Lemma 3.5. *Let P_1, \dots, P_n be n hyperplanes in \mathbb{R}^n passing the origin with linearly independent unit normal vectors ν_1, \dots, ν_n , and $V \subset \mathbb{R}^n$ be an infinite cone, with its vertex at the origin and a Lipschitz boundary, such that*

$$\partial V = \bigcup_{i=1}^n F_i,$$

where F_i is a subset of P_i with a nonempty interior. Suppose u is the unique nonnegative solution of (1.1)-(1.2) for $\Omega = V$. Then, for each $i = 1, \dots, n$ and any $x \in V$,

$$(3.5) \quad \text{dist}(x, F_i) |\partial_{\mu_i} u(x)| \leq C u(x),$$

where μ_i is a unit vector along $\bigcap_{j \neq i} F_j$, and C is a positive constant depending only on n and $\|(\nu_1, \dots, \nu_n)^{-1}\|$. Moreover, for any $x, x^* \in V$, if, for any $i = 1, \dots, n$,

$$(3.6) \quad |\langle x - x^*, \mu_i \rangle| \leq \tau |\langle x, \mu_i \rangle|,$$

for some constant $\tau > 0$, then

$$(3.7) \quad |u(x) - u(x^*)| \leq C \tau u(x).$$

Here and hereafter, $\|\cdot\|$ is the norm of matrices, considered as transforms in the Euclidean spaces. We always treat ν_i as a column vector. Then, (ν_1, \dots, ν_n) is an invertible $n \times n$ matrix. In the statement of Lemma 3.5, each F_i is a face of V and each $\bigcap_{j \neq i} F_j$ is an edge transversal to F_i . Hence in (3.5), directional derivatives along edges are estimated in terms of the distances to the corresponding faces.

Proof. Without loss of generality, we assume $\langle \nu_i, \mu_i \rangle > 0$, for any $i = 1, \dots, n$. Let e_i be the unit vector along the x_i -axis, for each $i = 1, \dots, n$. Consider the linear transform E given by

$$E = (\mu_1, \dots, \mu_n)^{-1}.$$

Then, E transforms μ_i to e_i and P_i to the hyperplane $\{x_i = 0\}$. Set $\tilde{V} = EV$ and, for $\tilde{x} \in \tilde{V}$,

$$\tilde{u}(\tilde{x}) = u(E^{-1}\tilde{x}).$$

Under the transform $\tilde{x} = Ex$ with $x \in V$, we have

$$(3.8) \quad a_{ij} \tilde{u}_{\tilde{x}_i \tilde{x}_j} = \frac{1}{4} n(n-2) \tilde{u}^{\frac{n+2}{n-2}} \quad \text{in } \tilde{V},$$

where $(a_{ij}) = EE^T$. We also have, for $i = 1, \dots, n$,

$$(3.9) \quad d_i = c_i |\tilde{x}_i|,$$

where d_i is the distance to P_i and c_i is a constant satisfying

$$\frac{1}{\|E\|} \leq c_i \leq \|E^{-1}\|.$$

We first note $\|E^{-1}\| \leq \sqrt{n}$. Next, we claim

$$(3.10) \quad \|E\| \leq \sqrt{n} \|(\nu_1, \dots, \nu_n)^{-1}\|.$$

To prove this, we consider the $n \times n$ matrix N given by

$$N = (\nu_1, \dots, \nu_n).$$

Then, $\langle \nu_l, \mu_j \rangle = 0$ for $l \neq j$. As a consequence, for any $j = 1, \dots, n$,

$$(3.11) \quad N^T \mu_j = \langle \nu_j, \mu_j \rangle e_j,$$

and hence $\|N^{-1}\| \geq \langle \nu_{1,j}, \mu_{1,j} \rangle^{-1}$. By writing (3.11) in its matrix form

$$N^T E^{-1} = \text{diag}(\langle \nu_1, \mu_1 \rangle, \dots, \langle \nu_n, \mu_n \rangle),$$

we obtain

$$\|E\| \leq \|N^T\| \cdot \|N^{-1}\| \leq \sqrt{n} \|N^{-1}\|.$$

This is (3.10). Now, \tilde{V} is bounded by faces $\tilde{F}_1, \dots, \tilde{F}_n$, with each $\tilde{F}_i = EF_i$ on a hyperplane. Without loss of generality, we assume, for each $i = 1, \dots, n$,

$$\tilde{F}_i \subset \{\tilde{x}_i = 0\}.$$

Set, for any $\tilde{x} \in \tilde{V}$,

$$\tilde{d}_i = \text{dist}(\tilde{x}, \tilde{F}_i).$$

We will prove

$$(3.12) \quad |\tilde{u}_{\tilde{x}_i}| \leq C \frac{\tilde{u}}{\tilde{d}_i}.$$

Without loss of generality, we assume, for a fixed $\tilde{x} \in \tilde{V}$,

$$\tilde{d}_1 \leq \tilde{d}_2 \leq \dots \leq \tilde{d}_n.$$

We note $\tilde{d}_1 = \text{dist}(\tilde{x}, \partial\tilde{V})$.

Case 1. First, we prove (3.12) for $i = 1$. By Lemma 3.2 and Lemma 3.4, we have, for $i = 1, \dots, n$,

$$(3.13) \quad |\tilde{u}_{\tilde{x}_i}| \leq C \tilde{d}_1^{\frac{n-2}{2}-1} \leq C \frac{\tilde{u}}{\tilde{d}_1}.$$

In particular, we have (3.12) for $i = 1$.

Case 2. Next, we prove (3.12) for $i = 2$.

Case 2.1. We assume $\tilde{d}_2 \leq 8n\|E\|\tilde{d}_1$. Then, by (3.13),

$$|\tilde{u}_{\tilde{x}_2}| \leq C \frac{\tilde{u}}{\tilde{d}_1} \leq C \frac{\tilde{u}}{\tilde{d}_2}.$$

Case 2.2. We assume $\tilde{d}_2 > 8n\|E\|\tilde{d}_1$. Set $r = \tilde{d}_2/8$. Then,

$$B_{8r}(\tilde{x}) \cap \partial\tilde{V} \subseteq \{\tilde{x}_1 = 0\}.$$

Let \tilde{p} be the point on $\{\tilde{x}_1 = 0\}$ with the smallest distance to \tilde{x} . Then, $B_{4r}(\tilde{p}) \cap \partial\tilde{V} \subseteq \{\tilde{x}_1 = 0\}$, and $B_{4r}(\tilde{p}) \cap \partial\tilde{V} \cap \{\tilde{x}_i = 0\} = \emptyset$ for $i = 2, \dots, n$.

Set

$$\overline{u}(\overline{x}) = r^{\frac{n-2}{2}} \tilde{u}(\tilde{p} + r\overline{x}).$$

By Theorem 3.1, we have

$$(3.14) \quad |(c_1 \overline{x}_1)^{\frac{n-2}{2}} \overline{u} - 1| \leq C \overline{d},$$

where c_1 is as in (3.9). Although Theorem 3.1 is formulated for bounded domains, the proof only requires the existence of the interior and exterior tangent balls. We also point out that Theorem 3.1 holds for solutions of (1.1) and that \tilde{u} satisfies (3.8). Therefore, there is an extra factor c_1 in (3.14).

Set

$$\overline{w} = (c_1 \overline{x}_1)^{\frac{n-2}{2}} \overline{u} - 1.$$

We denote by \overline{V} the image of \tilde{V} under the transform $(\cdot - \tilde{p})/r$. Then, \overline{w} satisfies

$$a_{ij} \overline{w}_{\overline{x}_i \overline{x}_j} + 2a_{ij} \frac{[(c_1 \overline{x}_1)^{-\frac{n-2}{2}}]_{\overline{x}_i}}{(c_1 \overline{x}_1)^{-\frac{n-2}{2}}} \overline{w}_{\overline{x}_j} = \frac{1}{4} n(n-2) (c_1 \overline{x}_1)^{-2} [(1 + \overline{w})^{\frac{n+2}{n-2}} - 1 - \overline{w}],$$

in $B_4 \cap \overline{V}$. For any fixed $\overline{x} \in B_2 \cap \overline{V}$, we consider the above equation in $B_{d_{\overline{x}}/2}(\overline{x})$. First, (3.14) implies

$$|\overline{w}| \leq C d_{\overline{x}} \quad \text{in } B_{d_{\overline{x}}/2}(\overline{x}).$$

Next, we have

$$\left| d_{\overline{x}} a_{ij} \frac{[(c_1 \overline{x}_1)^{-\frac{n-2}{2}}]_{\overline{x}_i}}{(c_1 \overline{x}_1)^{-\frac{n-2}{2}}} \right| \leq C \quad \text{in } B_{d_{\overline{x}}/2}(\overline{x}),$$

and

$$\left| d_{\overline{x}}^2 (c_1 \overline{x}_1)^{-2} [(1 + \overline{w})^{\frac{n+2}{n-2}} - 1 - \overline{w}] \right| \leq C d_{\overline{x}} \quad \text{in } B_{d_{\overline{x}}/2}(\overline{x}).$$

By applying the scaled interior estimate in $B_{d_{\bar{x}}/2}(\bar{x})$, we get, for $i = 1, \dots, n$,

$$|\bar{w}_{\bar{x}_i}(\bar{x})| \leq C,$$

and then, for $i = 2, \dots, n$,

$$|\bar{u}_{\bar{x}_i}| = |[(1 + \bar{w})(c\bar{x}_1)^{-\frac{n-2}{2}}]_{\bar{x}_i}| = |\bar{w}_{\bar{x}_i}(c\bar{x}_1)^{-\frac{n-2}{2}}| \leq C\bar{u},$$

where we used Lemma 3.2. Therefore, for $i = 2, \dots, n$,

$$(3.15) \quad |\tilde{u}_{\tilde{x}_i}| = |[r^{-\frac{n-2}{2}}\tilde{u}(\frac{\tilde{x} - \tilde{p}}{r})]_{\tilde{x}_i}| \leq C\frac{\tilde{u}}{r} \leq C\frac{u}{d_2}.$$

Hence, we have (3.12) for $i = 2$.

Case 3. Next, we prove (3.12) for $i = 3$.

Case 3.1. We assume $\tilde{d}_3 \leq 8n\|E\|\tilde{d}_2$, $\tilde{d}_2 \leq 8n\|E\|\tilde{d}_1$. Then, by (3.13),

$$|\tilde{u}_{\tilde{x}_3}| \leq C\frac{\tilde{u}}{\tilde{d}_1} \leq C\frac{\tilde{u}}{\tilde{d}_3}.$$

Case 3.2. We assume $\tilde{d}_3 \leq 8n\|E\|\tilde{d}_2$, $\tilde{d}_2 > 8n\|E\|\tilde{d}_1$. Then, by (3.15), we have

$$|\tilde{u}_{\tilde{x}_3}| \leq C\frac{u}{\tilde{d}_2} \leq C\frac{u}{\tilde{d}_3}.$$

Case 3.3. We assume $\tilde{d}_3 > 8n\|E\|\tilde{d}_2$. We first consider the case $n = 3$. For the given $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and any t sufficiently small, consider $\tilde{x}' = (\tilde{x}_1, \tilde{x}_2, (1+t)\tilde{x}_3)$ and $\tilde{x}'' = (1+t)\tilde{x}$. The mean-value theorem implies

$$|\tilde{u}(\tilde{x}') - \tilde{u}(\tilde{x}'')| \leq |\tilde{u}_{\tilde{x}_1}(\tilde{\xi}_1)t\tilde{x}_1| + |\tilde{u}_{\tilde{x}_2}(\tilde{\xi}_2)t\tilde{x}_2|,$$

where $\tilde{\xi}_1, \tilde{\xi}_2$ are points on the line segment between \tilde{x}' and \tilde{x}'' . By what we proved in Case 1 and Case 2, we have

$$|\tilde{u}(\tilde{x}') - \tilde{u}(\tilde{x}'')| \leq C \left(\frac{\tilde{u}(\tilde{\xi}_1)}{\tilde{d}_1(\tilde{\xi}_1)}|t\tilde{x}_1| + \frac{\tilde{u}(\tilde{\xi}_2)}{\tilde{d}_2(\tilde{\xi}_2)}|t\tilde{x}_2| \right) \leq C|t|\tilde{u}(\tilde{x}).$$

Next, by the scaling property (2.3) of solutions in cones, we get

$$|\tilde{u}(\tilde{x}) - \tilde{u}(\tilde{x}'')| = |(1+t)^{-\frac{n-2}{2}} - 1|\tilde{u}(\tilde{x}) \leq C|t|\tilde{u}(\tilde{x}),$$

and hence

$$|\tilde{u}(\tilde{x}) - \tilde{u}(\tilde{x}')| \leq C|t|\tilde{u}(\tilde{x}).$$

Dividing by $|t\tilde{x}_3|$ and letting $t \rightarrow 0$, we obtain

$$|\tilde{u}_{\tilde{x}_3}(\tilde{x})| \leq C\frac{\tilde{u}(\tilde{x})}{|\tilde{x}_3|}.$$

Note

$$\tilde{d}_3^2 \leq \tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 \leq \tilde{d}_1^2 + \tilde{d}_2^2 + \tilde{x}_3^2.$$

By the assumptions on $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3$, we obtain $\tilde{d}_3 \leq 2|\tilde{x}_3|$ and hence

$$|\tilde{u}_{\tilde{x}_3}(\tilde{x})| \leq C\frac{\tilde{u}(\tilde{x})}{\tilde{d}_3}.$$

Next, we consider the case $n \geq 4$. Set $r = \tilde{d}_3/8$. Then,

$$B_{8r}(\tilde{x}) \cap \partial\tilde{V} \subseteq \{\tilde{x}_1 = 0\} \cup \{\tilde{x}_2 = 0\}.$$

Let \tilde{V}_{12} be the infinite cone bounded by $\{\tilde{x}_1 = 0\}$ and $\{\tilde{x}_2 = 0\}$ such that $B_{8r}(\tilde{x}) \cap \tilde{V} = B_{8r}(\tilde{x}) \cap \tilde{V}_{12}$. Set \tilde{p} to be the point on $\partial\tilde{V}_{12}$ with the smallest distance to \tilde{x} . Then, $B_{4r}(\tilde{p}) \cap \partial\tilde{V} \subseteq \partial\tilde{V}_{12}$. Let \tilde{v} be the nonnegative solution of (3.8) and (1.2) for $\Omega = \tilde{V}_{12}$, which is a function of only two variables \tilde{x}_1 and \tilde{x}_2 .

Set

$$\bar{u}(\bar{x}) = r^{\frac{n-2}{2}} \tilde{u}(\tilde{p} + r\bar{x}),$$

and

$$\bar{v}(\bar{x}) = r^{\frac{n-2}{2}} \tilde{v}(\tilde{p} + r\bar{x}).$$

We denote by \bar{V} and \bar{V}_{12} the images of \tilde{V} and \tilde{V}_{12} under the transform $(\cdot - \tilde{p})/r$, respectively. Set

$$\bar{w} = \frac{\bar{u}}{\bar{v}} - 1.$$

Then, \bar{w} satisfies

$$a_{ij}\bar{w}_{\bar{x}_i\bar{x}_j} + 2a_{ij}\frac{\bar{v}_{\bar{x}_i}}{\bar{v}}\bar{w}_{\bar{x}_j} = \frac{1}{4}n(n-2)\bar{v}^{\frac{4}{n-2}}[(1+\bar{w})^{\frac{n+2}{n-2}} - 1 - \bar{w}] \quad \text{in } B_4 \cap \bar{V}_{12}.$$

Next, we claim

$$(3.16) \quad |\bar{w}| \leq C\bar{d}^{\frac{n-2}{2}} \quad \text{in } B_2 \cap \bar{V}.$$

For any fixed $\bar{x} \in B_2 \cap \bar{V}$, we note $\bar{V} \cap B_{1/2}(\bar{x}) \subset \bar{V}_{12}$ and $\bar{V}_{12} \cap B_{1/2}(\bar{x}) \subset \bar{V}$. Let $u_{\bar{x},1/2}$ be the solution of (1.1)-(1.2) in $B_{1/2}(\bar{x})$, given by (2.1). Lemma 2.2 implies

$$\bar{u} \leq \bar{v} + u_{\bar{x},1/2}, \quad \bar{u} \leq \bar{v} + u_{\bar{x},1/2} \quad \text{in } B_{1/2}(\bar{x}).$$

Note $u_{\bar{x},1/2}(\bar{x}) \leq C$. By Lemma 3.2, we have

$$\begin{aligned} \bar{u}(\bar{x}) &\leq \bar{v}(\bar{x}) + C \leq \bar{v}(\bar{x})(1 + C\bar{d}^{\frac{n-2}{2}}), \\ \bar{v}(\bar{x}) &\leq \bar{u}(\bar{x}) + C \leq \bar{u}(\bar{x})(1 + C\bar{d}^{\frac{n-2}{2}}). \end{aligned}$$

This finishes the proof of (3.16). By $n \geq 4$, we get

$$(3.17) \quad |\bar{w}| \leq C\bar{d} \quad \text{in } B_2 \cap \bar{V}.$$

For any fixed $\bar{x} \in B_2 \cap \bar{V}$, we have

$$|\bar{w}| \leq C\bar{d}_{\bar{x}} \quad \text{in } B_{\bar{d}_{\bar{x}}/2}(\bar{x}).$$

Moreover,

$$\left| d_{\bar{x}} a_{ij} \frac{\bar{v}_{\bar{x}_j}}{\bar{v}} \right| \leq C \quad \text{in } B_{\bar{d}_{\bar{x}}/2}(\bar{x}),$$

and

$$d_{\bar{x}}^2 \bar{v}^{\frac{4}{n-2}} |(1+\bar{w})^{\frac{n+2}{n-2}} - 1 - \bar{w}| \leq C\bar{d}_{\bar{x}} \quad \text{in } B_{\bar{d}_{\bar{x}}/2}(\bar{x}).$$

By applying the scaled interior estimate in $B_{\bar{d}_{\bar{x}}/2}(\bar{x})$, we have, for $i = 1, \dots, n$,

$$|\bar{w}_{\bar{x}_i}(\bar{x})| \leq C,$$

and then, for $i = 3, \dots, n$,

$$|\overline{u}_{\overline{x}_i}| = |[(1 + \overline{w})\overline{v}]_{\overline{x}_i}| = |\overline{w}_{\overline{x}_i}\overline{v}| \leq C\overline{u},$$

where we used Lemma 3.2 and the fact that \overline{v} is a function of \overline{x}_1 and \overline{x}_2 . Therefore, for $i = 3, \dots, n$,

$$|\tilde{u}_{\tilde{x}_i}| = |[r^{-\frac{n-2}{2}}\tilde{u}(\frac{\tilde{x} - \tilde{p}}{r})]_{\tilde{x}_i}| \leq C\frac{\tilde{u}}{r} \leq C\frac{\tilde{u}}{\tilde{d}_3}.$$

Similarly, we can prove (3.12) for general i . This is (3.5) in \tilde{V} .

Next, we prove (3.7). In \tilde{V} , (3.6) reduces to

$$|\tilde{x}_i - \tilde{x}_i^*| \leq \tau|\tilde{x}_i|.$$

By (3.12) and the fact $|\tilde{x}_i| \leq \tilde{d}_i$, we have

$$|\tilde{u}(\tilde{x}) - \tilde{u}(\tilde{x}^*)| \leq C\tau\tilde{u}(\tilde{x}).$$

This is (3.7) in \tilde{V} . □

Lemma 3.5, appropriately modified, holds if the boundary ∂V is formed from k linearly independent hyperplanes, for some $k < n$. In this case, we can assume $V = V_k \times \mathbb{R}^{n-k}$, where V_k is an infinite cone in \mathbb{R}^k , and then by Lemma 2.3, the solution u is a function of $(x_1, \dots, x_k) \in \mathbb{R}^k$.

4. DOMAINS AND THEIR TANGENT CONES

In this section, we discuss bounded Lipchitz domains whose boundaries consist of finitely many $C^{1,1}$ -hypersurfaces locally and focus on the relation between these domains and their tangent cones. To do this, we will place these domains in a good position and express boundaries of the Lipchitz domains and boundaries of the tangent cones by functions satisfying the same algebraic relation. At the first glance, this is a tedious way to describe such a relation and does not seem necessary. As mentioned in the introduction, an important step in the derivation of the asymptotic expansion in the proof of Theorem 1.1 is to construct two sets, one inside the domains and another containing the domains. The algebraic relation governing the relation between the domains and their tangent cones will provide an easy description to construct these two sets. As we will see, the same algebraic relation determines four sets, the domains bounded by k $C^{1,1}$ -hypersurfaces, their tangent cones bounded by k hyperplanes, the sets inside the domains bounded by k spheres, and the sets containing the domains also bounded by k spheres.

We start our discussion with infinite cones, which serve as tangent cones. We emphasize that all cones in this paper are solid.

Any finitely many hyperplanes passing the origin divide \mathbb{R}^n into finitely many connected components. Each component is a cone. In the following, we discuss unions of these components.

We first introduce the concept of signed distances. Let P be a hyperplane with a unit normal vector ν and p be a point on P . Then, the signed distance of x with respect to ν is defined by

$$(4.1) \quad d(x) = \begin{cases} |\text{dist}(x, P)| & \text{if } \langle x - p, \nu \rangle > 0, \\ 0 & \text{if } x \in P, \\ -|\text{dist}(x, P)| & \text{if } \langle x - p, \nu \rangle < 0. \end{cases}$$

Obviously, the signed distance is independent of the choice of $p \in P$.

Fix an integer $k \geq 2$.

Definition 4.1. Let P_1, \dots, P_k be k hyperplanes in \mathbb{R}^n passing the origin with mutually distinct normal vectors and let V be a Lipschitz infinite cone. Then, V is called to be bounded by P_1, \dots, P_k if

$$\partial V \subseteq \bigcup_{i=1}^k P_i.$$

We call $F_i = \partial V \cap P_i$ a face of V . For convenience, we always assume $\overline{\partial V \setminus P_i} \neq \partial V$, for each $i = 1, \dots, k$.

Next, we put the hyperplanes P_1, \dots, P_k and the cone V as in Definition 4.1 in a standard position. We choose a coordinate system such that, for each $i = 1, \dots, k$,

$$P_i = \{x \in \mathbb{R}^n : x_n = L_i(x')\},$$

for some linear function L_i in \mathbb{R}^{n-1} , and

$$V = \{x \in \mathbb{R}^n : x_n > g(x')\},$$

for some $g(x') \in \{L_1(x'), L_2(x'), \dots, L_k(x')\}$. Then,

$$\partial V = \{x \in \mathbb{R}^n : x_n = g(x')\}.$$

Next, let ν_1, \dots, ν_k be unit normal vectors of P_1, \dots, P_k , respectively, such that, for any $i = 1, \dots, k$,

$$(4.2) \quad \langle \nu_i, e_n \rangle > 0.$$

The vectors ν_1, \dots, ν_k are called the *inner* unit normal vectors associated with the cone V . Moreover, set

$$\begin{aligned} H_i^1 &= \{x \in \mathbb{R}^n : x_n > L_i(x')\}, \\ H_i^{-1} &= \{x \in \mathbb{R}^n : x_n \leq L_i(x')\}, \end{aligned}$$

and

$$(4.3) \quad V_{(l_1, \dots, l_k)} = \bigcap_{i=1}^k H_i^{l_i},$$

where $l_i = 1$ or -1 for each $i = 1, \dots, k$. Then, the Lipschitz infinite cone V as in Definition 4.1 can be expressed by the union of some $V_{(l_1, \dots, l_k)}$, i.e.,

$$(4.4) \quad V = \bigcup V_{(l_1, \dots, l_k)},$$

where the union is over a finite collection of vectors of the form (l_1, \dots, l_k) , with $l_i = 1$ or -1 for each $i = 1, \dots, k$.

It is straightforward to verify the following result.

Lemma 4.2. *Let V be a Lipschitz infinite cone bounded by k hyperplanes P_1, \dots, P_k with mutually distinct normal vectors in the above setting. Then,*

$$(i) \quad V_{(\underbrace{1, \dots, 1}_k)} \subseteq V \text{ and } V_{(\underbrace{-1, \dots, -1}_k)} \cap V = \emptyset;$$

$$(ii) \text{ if } (l_1, \dots, l_k) \leq (m_1, \dots, m_k) \text{ and } V_{(l_1, \dots, l_k)} \subseteq V, \text{ then, } V_{(m_1, \dots, m_k)} \subseteq V.$$

Here and hereafter, $(l_1, \dots, l_k) \leq (m_1, \dots, m_k)$ simply means $l_i \leq m_i$ for each $i = 1, \dots, k$.

We point out that the interior of $\partial V \cap P_i$ may have more than one connected components, even for $k \leq n$.

Example 4.3. Let P_1, P_2, P_3 be 3 linearly independent hyperplanes in \mathbb{R}^3 satisfying (4.2) for $n = 3$ and $V \subseteq \mathbb{R}^3$ be an infinite cone given by

$$V = V_{(1,1,1)} \bigcup V_{(-1,1,1)} \bigcup V_{(1,-1,1)} \bigcup V_{(1,1,-1)}.$$

The interior of each $\partial V \cap P_i$ has two connected components.

Definition 4.4. Let $V_1, V_2 \subseteq \mathbb{R}^n$ be two infinite cones bounded by two sets of hyperplanes $P_{1,1}, \dots, P_{1,k}$ and $P_{2,1}, \dots, P_{2,k}$, respectively. Then, V_1 and V_2 are said to *satisfy the same relation* if the collections of $(l_{1,1}, \dots, l_{1,k})$ in (4.4) for V_1 and of $(l_{2,1}, \dots, l_{2,k})$ in (4.4) for V_2 are identical.

Let $V \subseteq \mathbb{R}^n$ be an infinite cone as in Definition 4.1 and ν_1, \dots, ν_k be its inner unit normal vectors. Now, we require $k \leq n$ and ν_1, \dots, ν_k are linearly independent. By a rotation, we assume $V = V_k \times \mathbb{R}^{n-k}$, with $V_k \subset \mathbb{R}^k$. Then for any x , the projection from x to $\mathbb{R}^k \times \{0\}$ can be uniquely determined by $d_1(x), \dots, d_k(x)$, where d_i is the signed distance from x to P_i with respect to ν_i . With such a one-to-one correspondence between $x \in \mathbb{R}^k \times \{0\}$ and (d_1, \dots, d_k) , we rewrite the solution of (1.1)-(1.2) for $\Omega = V$ in Theorem 2.3 as

$$(4.5) \quad f_V(d_1(x), \dots, d_k(x)) = u_V(x).$$

If we treat ν_1, \dots, ν_k as column vectors, then the matrix (ν_1, \dots, ν_k) is a $k \times n$ matrix. By the linear independence, the $k \times k$ matrix $(\nu_1, \dots, \nu_k)^T (\nu_1, \dots, \nu_k)$ is invertible. Set

$$(4.6) \quad \sigma(P_1, \dots, P_k) = \|((\nu_1, \dots, \nu_k)^T (\nu_1, \dots, \nu_k))^{-1}\|.$$

We also note

$$\{d_1 > 0, \dots, d_k > 0\} \subset V, \quad \{d_1 < 0, \dots, d_k < 0\} \cap V = \emptyset.$$

For $k = n$, $\bigcap_{j \neq i} P_j$ is an *edge* of V , which is transversal to P_i . In the following, we always denote by μ_i the unique unit vector such that

$$(4.7) \quad \mu_i \in \bigcap_{j \neq i} P_j, \quad \langle \mu_i, \nu_i \rangle > 0.$$

Then, we can check

$$\{d_1 > 0, \dots, d_n > 0\} = \{t_1 \mu_1 + \dots + t_n \mu_n : t_1 > 0, \dots, t_n > 0\},$$

and

$$\{d_1 < 0, \dots, d_n < 0\} = \{t_1 \mu_1 + \dots + t_n \mu_n : t_1 < 0, \dots, t_n < 0\}.$$

In Example 4.3, edges consist of three lines, instead of three rays we usually anticipate.

Next, we turn our attention to domains. We always assume that Ω is a bounded Lipschitz domain, with $x_0 \in \partial\Omega$. We can define the *tangent cone* of Ω at x_0 by blowing up Ω near x_0 . We will not present such a definition for general domains. In the following, we describe an equivalent way to construct tangent cones for domains bounded by finitely many $C^{1,1}$ -hypersurfaces.

Let S be a $C^{1,1}$ -hypersurface in a neighborhood of $x_0 \in S$, and ν be a continuous unit normal vector field over S near x_0 . Here and hereafter, we always assume x_0 is an interior point of S . For any x close to x_0 , set p_x to be the point on S with the least distance to x . Then, the *signed distance* of x to S with respect to ν is defined by

$$(4.8) \quad d(x) = \begin{cases} |xp_x| & \text{if } \langle x - p_x, \nu_{p_x} \rangle > 0, \\ 0 & \text{if } x \in S, \\ -|xp_x| & \text{if } \langle x - p_x, \nu_{p_x} \rangle < 0. \end{cases}$$

Fix an integer $k \geq 2$.

Definition 4.5. Let S_1, \dots, S_k be k $C^{1,1}$ -hypersurfaces passing x_0 , with mutually distinct normal vectors of tangent planes of S_1, \dots, S_k at x_0 , and let Ω be a bounded Lipschitz domain in \mathbb{R}^n with $x_0 \in \partial\Omega$. Then, Ω is called to be bounded by S_1, \dots, S_k near x_0 if, for some $R > 0$,

$$\partial\Omega \cap B_R(x_0) \subseteq \bigcup_{i=1}^k S_i.$$

We always assume $\overline{\partial\Omega \cap B_r(x_0) \setminus S_i} \neq \partial\Omega \cap B_r(x_0)$, for any $r \leq R$ and any $i = 1, \dots, k$.

Let S_1, \dots, S_k and Ω be as in Definition 4.5. We choose a coordinate system such that, for each $i = 1, \dots, k$,

$$S_i \cap B_R(x_0) = \{x \in B_R(x_0) : x_n = f_i(x')\},$$

for some $C^{1,1}$ -function f_i in $B'_R(x'_0)$, and

$$\Omega \cap B_R(x_0) = \{x \in B_R(x_0) : x_n > g(x')\},$$

for some $g(x') \in \{f_1(x'), f_2(x'), \dots, f_k(x')\}$. Then,

$$\partial\Omega \cap B_R(x_0) = \{x \in B_R(x_0) : x_n = g(x')\}.$$

Next, let ν_1, \dots, ν_k be unit normal vectors of the tangent planes of S_1, \dots, S_k at x_0 such that, for any $i = 1, \dots, k$,

$$\langle \nu_i, e_n \rangle > 0.$$

The vectors ν_1, \dots, ν_k are called the *inner* unit normal vectors associated with $\partial\Omega$ at x_0 . Moreover, set

$$\begin{aligned} S_i^1 &= \{x \in B_R(x_0) : x_n > f_i(x')\}, \\ S_i^{-1} &= \{x \in B_R(x_0) : x_n \leq f_i(x')\}, \end{aligned}$$

and

$$\Omega_{(l_1, \dots, l_k)} = \bigcap_{i=1}^k S_i^{l_i},$$

where $l_i = 1$ or -1 for each $i = 1, \dots, k$. Then, $\Omega \cap B_R(x_0)$ can be expressed by the union of some $\Omega_{(l_1, \dots, l_k)}$, i.e.,

$$(4.9) \quad \Omega \cap B_R(x_0) = \bigcup \Omega_{(l_1, \dots, l_k)},$$

where the union is over a finite collection of vectors of the form (l_1, \dots, l_k) , with $l_i = 1$ or -1 for each $i = 1, \dots, k$.

It is straightforward to verify the following result.

Lemma 4.6. *Let Ω be a bounded Lipschitz domain bounded by k $C^{1,1}$ -hypersurfaces S_1, \dots, S_k in $B_R(x_0)$ for some $x_0 \in \partial\Omega$ and some $R > 0$ in the above setting, with mutually distinct normal vectors of the tangent planes of S_1, \dots, S_k at x_0 . Then,*

- (i) $\Omega_{(\underbrace{1, \dots, 1}_k)} \subseteq \Omega$ and $\Omega_{(\underbrace{-1, \dots, -1}_k)} \cap \Omega = \emptyset$;
- (ii) if $(l_1, \dots, l_k) \leq (m_1, \dots, m_k)$ and $\Omega_{(l_1, \dots, l_k)} \subseteq \Omega$, then, $\Omega_{(m_1, \dots, m_k)} \subseteq \Omega$.

For domains as in Definition 4.5, we can characterize their tangent cones easily. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain bounded by k $C^{1,1}$ -hyperplanes S_1, \dots, S_k in a neighborhood of $x_0 \in \partial\Omega$, satisfying (4.9). Suppose that the tangent plane P_i of S_i at x_0 is given by $x_n = L_i(x')$, for each $i = 1, \dots, k$. Then, the tangent cone V_{x_0} of Ω at x_0 is the infinite cone given by

$$(4.10) \quad V_{x_0} = \bigcup V_{(l_1, \dots, l_k)},$$

where $V_{(l_1, \dots, l_k)}$ is defined in (4.3), and the union over (l_1, \dots, l_k) in (4.10) is the same as that in (4.9).

To end this section, we make a remark concerning the necessity to put cones and domains in standard positions. Recall that we start with a bounded Lipschitz domain which is bounded by finitely many $C^{1,1}$ -hypersurfaces near a boundary point. By putting this domain in the standard position, we can characterize its tangent cone easily, as demonstrated in the proceeding paragraph. The advantage here is to simply take the union over the same set of finitely elements in the form (l_1, \dots, l_k) , with $l_i = 1$ or -1 for each $i = 1, \dots, k$. In Section 6, we will construct more sets bounded by spheres along the same line.

5. SOLUTIONS IN DIFFERENT CONES

In section, we compare solutions in different cones.

We first prove a basic result. We point out that, for an infinite cone with a Lipschitz boundary, there always exists a uniform infinite exterior cone at each point of its boundary.

Lemma 5.1. *Let $V_1, V_2 \subseteq \mathbb{R}^n$ be two infinite cones with Lipschitz boundaries and vertices at the origin and u_i be the nonnegative solution of (1.1)-(1.2) for $\Omega = V_i$, $i = 1, 2$. Assume there exists a linear transform $T = O_1 A O_2$, for some $O_1, O_2 \in O(n)$ and $A \in GL(n)$, such that $TV_1 = V_2$. Then, there exist a positive constant ϵ_0 , depending only on n and the size of exterior cones, such that, if $\|A - I\| \leq \epsilon_0$, then for any $x \in V_2$,*

$$(1 + C\|A - I\|)^{-1}u_2(x) \leq u_1(T^{-1}x) \leq (1 + C\|A - I\|)u_2(x),$$

where C is a positive constant depending only on n and the size of exterior cones.

As noted before, $\|\cdot\|$ is the norm of matrices, considered as transforms in the Euclidean spaces.

Proof. First, u_1 satisfies

$$\Delta u_1 = \frac{1}{4}n(n-2)u_1^{\frac{n+2}{n-2}} \quad \text{in } V_1.$$

Set $\tilde{u}_1(x) = u_1(T^{-1}x)$, for any $x \in V_2$, and $(a_{ij}) = AA^T$. Then,

$$a_{ij}\tilde{u}_1 = \frac{1}{4}n(n-2)\tilde{u}_1^{\frac{n+2}{n-2}} \quad \text{in } V_2,$$

and hence

$$-\Delta\tilde{u}_1 + \frac{1}{4}n(n-2)\tilde{u}_1^{\frac{n+2}{n-2}} = (a_{ij} - \delta_{ij})\tilde{u}_1 \quad \text{in } V_2.$$

Therefore,

$$\begin{aligned} & -\Delta[(1 + C\|A - I\|)\tilde{u}_1] + \frac{1}{4}n(n-2)[(1 + C\|A - I\|)\tilde{u}_1]^{\frac{n+2}{n-2}} \\ &= \frac{1}{4}n(n-2)[(1 + C\|A - I\|)^{\frac{n+2}{n-2}} - (1 + C\|A - I\|)]\tilde{u}_1^{\frac{n+2}{n-2}} \\ & \quad + (1 + C\|A - I\|)(a_{ij} - \delta_{ij})\tilde{u}_1. \end{aligned}$$

We first choose $\epsilon_0 < 1/10$. Then,

$$\|A^{-1}\| = \|(I - (I - A))^{-1}\| \leq \frac{1}{1 - \|A - I\|},$$

and

$$(1 - \|A - I\|)\text{dist}(x, \partial V_2) \leq \text{dist}(T^{-1}x, \partial V_1) \leq (1 + \|A - I\|)\text{dist}(x, \partial V_2).$$

Next, we note

$$|\delta_{ij} - a_{ij}| \leq \|A^T A - I\| \leq 3\|A - I\|,$$

and, by Lemma 3.2 and Remark 3.3,

$$C' \leq \text{dist}(x, \partial V_i)^{\frac{n-2}{2}} u_i(x) \leq 2^{\frac{n-2}{2}}.$$

Lemma 3.4 implies

$$|(\delta_{ij} - a_{ij})\tilde{u}_{1ij}| \leq C_1 \|A - I\| \text{dist}(x, \partial V_2)^{-\frac{n+2}{2}}.$$

We also have

$$\begin{aligned} & [(1 + C\|A - I\|)^{\frac{n+2}{n-2}} - (1 + C\|A - I\|)] \tilde{u}_1^{\frac{n+2}{n-2}} \\ & \geq \frac{4}{n-2} C\|A - I\| \tilde{u}_1^{\frac{n+2}{n-2}} \geq C\|A - I\| C_2(n) \text{dist}(x, \partial V_2)^{-\frac{n+2}{2}}. \end{aligned}$$

By taking $C = C_3 > 2C_1/C_2$ and requiring $\epsilon_0 \leq 1/C_3$, we obtain

$$-\Delta[(1 + C\|A - I\|)\tilde{u}_1] + \frac{1}{4}n(n-2)[(1 + C\|A - I\|)\tilde{u}_1]^{\frac{n+2}{n-2}} \geq 0.$$

Therefore, by the maximum principle, we get

$$u_2(x) \leq u_1(T^{-1}x)(1 + C\|A - I\|).$$

Interchanging the position of u_1 and u_2 , we also have

$$u_1(T^{-1}x) \leq (1 + C\|A^{-1} - I\|)u_2(x) \leq (1 + C\|A - I\|)u_2(x).$$

In summary, by choosing $\epsilon_0 \leq \min\{1/10, 1/C_3\}$, we get the desired estimate. \square

Next, we discuss cones introduced in Definition 4.1 and compare different f_V introduced in (4.5) in different cones. Lemma 3.5 plays an essential role.

Lemma 5.2. *For a fixed $2 \leq k \leq n$, let $V_1, V_2 \subseteq \mathbb{R}^n$ be two infinite cones satisfying the same relation as in Definition 4.4 in terms of linearly independent hyperplanes $P_{1,1}, \dots, P_{1,k}$ and $P_{2,1}, \dots, P_{2,k}$, respectively, and u_i be the unique nonnegative solution of (1.1)-(1.2) for $\Omega = V_i$, with f_{V_i} given by (4.5), for $i = 1, 2$. Assume there exists a linear transform $T = O_1 A O_2$, for some $O_1, O_2 \in O(n)$ and $A \in GL(n)$, such that $TV_1 = V_2$ and $T(\partial V_1 \cap P_{1,j}) = \partial V_2 \cap P_{2,j}$, for $j = 1, \dots, k$. For the positive constant ϵ_0 determined in Lemma 5.1, if $\|A - I\| \leq \epsilon_0$, then*

$$(1 + C\|A - I\|)^{-1} f_{V_2}(d_1, \dots, d_k) \leq f_{V_1}(d_1, \dots, d_k) \leq (1 + C\|A - I\|) f_{V_2}(d_1, \dots, d_k),$$

where C is some positive constant depending only on n and $\sigma(P_{i,1}, \dots, P_{i,k})$ as defined in (4.6).

Proof. We consider only the case $k = n$. Fix d_1, d_2, \dots, d_n such that the corresponding $x \in V_2$. By Lemma 5.1, we have

$$(5.1) \quad (1 + C\|A - I\|)^{-1} u_2(x) \leq u_1(T^{-1}x) \leq (1 + C\|A - I\|) u_2(x).$$

Note, for each $j = 1, \dots, n$,

$$|\hat{d}_j - d_j| \leq \|A - I\| |d_j|,$$

where \hat{d}_j is the signed distance from $T^{-1}x$ to $P_{1,j}$. Let x^* be the point in V_1 such that its signed distance to $P_{1,j}$ is d_j . Then,

$$|(T^{-1}x)_i - x_i^*| \leq C\|A - I\| |x_i^*|.$$

By Lemma 3.5, in particular (3.7), we have

$$(5.2) \quad |u_1(T^{-1}x) - u_1(x^*)| \leq C\|A - I\|u_1(x^*).$$

By combining with (5.1) and (5.2), we obtain

$$(1 + C\|A - I\|)^{-1}u_2(x) \leq u_1(x^*) \leq (1 + C\|A - I\|)u_2(x).$$

This implies the desired result. \square

Next, we express the condition in Lemma 5.2 in terms of the inner unit normal vectors.

Let V be an infinite cone as introduced in Definition 4.1 and ν_1, \dots, ν_k be inner unit normal vectors of V . Consider the $n \times k$ matrix

$$(5.3) \quad N = (\nu_1, \dots, \nu_k).$$

If $k = n$, let μ_1, \dots, μ_n be the vectors introduced in (4.7). Then, for any $j = 1, \dots, n$,

$$\langle \nu_l, \mu_j \rangle = 0 \quad \text{for } l \neq j,$$

and

$$\langle \nu_j, \mu_j \rangle > 0.$$

As a consequence, for any $j = 1, \dots, n$,

$$N^T \mu_j = \langle \nu_j, \mu_j \rangle e_j,$$

where e_j is the unit vector along the x_j -axis. This implies, in particular, that $N^T \mu_j$ and e_j , or μ_j and $N^{-T} e_j$, are along the same direction.

We have the following result.

Theorem 5.3. *For a fixed $2 \leq k \leq n$, let $V_1, V_2 \subseteq \mathbb{R}^n$ be two infinite cones satisfying the same relation as in Definition 4.4 in terms of linearly independent hyperplanes $P_{1,1}, \dots, P_{1,k}$ and $P_{2,1}, \dots, P_{2,k}$, respectively, and u_i be the unique nonnegative solution of (1.1)-(1.2) for $\Omega = V_i$, with f_{V_i} given by (4.5), for $i = 1, 2$. For the positive constant ϵ_0 determined in Lemma 5.1, if*

$$3n\|(N_1^T N_1)^{-1}\|\|N_1 - N_2\| \leq \epsilon_0,$$

then,

$$(1 + C\|N_1 - N_2\|)^{-1}f_{V_2}(d_1, \dots, d_k) \leq f_{V_1}(d_1, \dots, d_k) \leq (1 + C\|N_1 - N_2\|)f_{V_2}(d_1, \dots, d_k),$$

where N_1 and N_2 are the matrices defined as in (5.3) for V_1 and V_2 , and C is a positive constant depending only on n and $\sigma(P_{i,1}, \dots, P_{i,k})$ as defined in (4.6).

Proof. We first consider $k = n$. Set

$$(5.4) \quad A = (\mu_{2,1}, \dots, \mu_{2,n})(\mu_{1,1}, \dots, \mu_{1,n})^{-1}.$$

Then, the linear transform given by $T = A$ satisfies $T\mu_{1,j} = \mu_{2,j}$, for $j = 1, \dots, n$. Since V_1, V_2 satisfy the same relation, then $TV_1 = V_2$ and $TF_{1,j} = F_{2,j}$, for $j = 1, \dots, n$. We now verify $\|A - I\| \leq \epsilon_0$. First, we have

$$(5.5) \quad \mu_{i,j} = (N_i^T)^{-1} \langle \nu_{i,j}, \mu_{i,j} \rangle e_j.$$

Then,

$$\begin{aligned}
& |(N_1^T)^{-1}\langle \nu_{1,j}, \mu_{1,j} \rangle e_j - (N_2^T)^{-1}\langle \nu_{1,j}, \mu_{1,j} \rangle e_j| \\
& \leq \|(N_2^T)^{-1}\| \|N_2^T - N_1^T\| \|(N_1^T)^{-1}\langle \nu_{1,j}, \mu_{1,j} \rangle e_j| \\
& \leq \|N_2^{-1}\| \|N_2 - N_1\| \\
& \leq \frac{\|N_1^{-1}\|}{1 - \|N_1^{-1}\| \|N_2 - N_1\|} \|N_2 - N_1\|.
\end{aligned}$$

Therefore, if $\|N_1^{-1}\| \|N_2 - N_1\| \leq 1/10$, we have

$$\begin{aligned}
|\mu_{1,j} - \mu_{2,j}| & \leq |\mu_{1,j} - (N_2^T)^{-1}\langle \nu_{1,j}, \mu_{1,j} \rangle e_j| + |(N_2^T)^{-1}\langle \nu_{1,j}, \mu_{1,j} \rangle e_j - \mu_{2,j}| \\
& \leq 2|\mu_{1,j} - (N_2^T)^{-1}\langle \nu_{1,j}, \mu_{1,j} \rangle e_j| \\
& \leq 3\|N_1^{-1}\| \|N_2 - N_1\|,
\end{aligned}$$

where, for the estimate of $(N_2^T)^{-1}\langle \nu_{1,j}, \mu_{1,j} \rangle e_j - \mu_{2,j}$, we use the facts that $(N_2^T)^{-1}e_j$ and $\mu_{2,j}$ are on the same ray and $|\mu_{2,j}| = |\mu_{1,j}|$. Hence,

$$\begin{aligned}
\|A - I\| & \leq \|(\mu_{2,1} - \mu_{1,1}, \dots, \mu_{2,n} - \mu_{1,n})\| \|(\mu_{1,1}, \dots, \mu_{1,n})^{-1}\| \\
& \leq 3\sqrt{n}\|N_1^{-1}\| \|N_2 - N_1\| \|(\mu_{1,1}, \dots, \mu_{1,n})^{-1}\|.
\end{aligned}$$

By (3.10), we obtain

$$\|(\mu_{1,1}, \dots, \mu_{1,n})^{-1}\| \leq \sqrt{n}\|N_1^{-1}\|,$$

and then

$$\|A - I\| \leq 3n\|N_1^{-1}\|^2 \|N_2 - N_1\|.$$

Hence, $\|A - I\| \leq \epsilon_0$. Then, we can apply Lemma 5.2.

Next, we consider $2 \leq k < n$. Set $E = \text{span}\{\nu_{1,1}, \nu_{1,2}, \dots, \nu_{1,k}\}$ and let e_{k+1}, \dots, e_n be the orthonormal basis of the orthogonal complement of E . Consider the matrices

$$\overline{N}_1 = (\nu_{1,1}, \dots, \nu_{1,k}, e_{k+1}, \dots, e_n), \quad \overline{N}_2 = (\nu_{2,1}, \dots, \nu_{2,k}, e_{k+1}, \dots, e_n).$$

Then, we have

$$\|\overline{N}_1^{-1}\|^2 = \|(\overline{N}_1^T \overline{N}_1)^{-1}\| = \|(N_1^T N_1)^{-1}\|,$$

and

$$\|N_1 - N_2\| = \|\overline{N}_1 - \overline{N}_2\|.$$

Let P_j be the hyperplane passing the origin with its unit normal vector e_j , $j = k+1, \dots, n$. For a unification, we write $P_{i,j} = P_j$, for $i = 1, 2$ and $j = k+1, \dots, n$. Let \overline{V}_i be the *convex* infinite cone enclosed by the n hyperplanes $P_{i,1}, \dots, P_{i,k}, P_{i,k+1}, \dots, P_{i,n}$ and \widehat{V}_i be the *convex* infinite cone enclosed by the k hyperplanes $P_{i,1}, \dots, P_{i,k}$. Denote by $F_{i,1}, \dots, F_{i,n}$ the faces of \overline{V}_i and by $\mu_{i,j}$ the unit vector along $\bigcap_{l \neq j} F_{i,l}$. Then, for $i = 1, 2$,

$$\overline{V}_i = \{t_{i,1}\mu_{i,1} + \dots + t_{i,n}\mu_{i,n} : t_{i,j} > 0 \text{ for } j = 1, \dots, n\}.$$

Define A by (5.4). Hence, the affine transform given by $T = A$ satisfies $T\mu_{1,j} = \mu_{2,j}$, for $j = 1, \dots, n$, and therefore $TV_1 = V_2$. By Lemma 5.2, we have the desired estimate. \square

6. SOLUTIONS NEAR SINGULAR POINTS

In this section, we study the asymptotic expansions near singular points intersected by $C^{1,1}$ -hypersurfaces in \mathbb{R}^n and derive an optimal estimate. The discussion relies essentially on the conformal invariance of the equation (1.1).

We first prove some results concerning the intersection of spheres.

Lemma 6.1. *Let $\partial B_{R_i}(o_i)$ be two circles in \mathbb{R}^2 , $i = 1, 2$, intersecting at two points p and q . Suppose the angle between po_1 and po_2 is α for some $\alpha \in (0, \pi)$. Then,*

$$|pq| = \frac{2R_1 R_2 \sin \alpha}{\sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \cos \alpha}}.$$

Proof. This follows by a straightforward calculation. We simply note $o_1 o_2 \perp pq$ and $|o_1 o_2| = \sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \cos \alpha}$. \square

Lemma 6.2. *Let $\partial B_1(o_i)$ be n unit spheres in \mathbb{R}^n , $i = 1, \dots, n$, intersecting at a point p , and the inner unit normal vector ν_i of $\partial B_1(o_i)$ at p be linearly independent. Then, $\partial B_1(o_i)$ intersect at another point q , and*

$$|pq| > \frac{|\det N|}{2^{n-2}},$$

where N is the matrix $(\nu_1, \nu_2, \dots, \nu_n)$.

Proof. We prove by induction.

For $n = 2$, by Lemma 6.1, $\partial B_1(o_i)$ intersect at two points p and q , and

$$|pq| = \frac{2 \sin \alpha}{\sqrt{2 - 2 \cos \alpha}} > \sin \alpha = |\det N|,$$

where α is the angle between ν_1 and ν_2 . Hence, the desired result holds for $n = 2$. In fact, these two circles intersect at exactly two points.

Now suppose the result holds for $n = k$, for some $k \geq 2$, and we consider $k + 1$. Without loss of generality, we assume $p = 0$ and

$$\begin{aligned} \nu_i &= (\nu_1^i, \dots, \nu_k^i, 0)^T \quad \text{for } i = 1, \dots, k, \\ \nu_{k+1} &= (\nu_1^{k+1}, \dots, \nu_k^{k+1}, \nu_{k+1}^{k+1})^T \quad \text{with } \nu_{k+1}^{k+1} > 0. \end{aligned}$$

Under this setting, the $(k + 1)$ -th coordinate of o_i is zero, for $i = 1, \dots, k$. Write $\nu'_i = (\nu_1^i, \dots, \nu_k^i)^T$ and $N_k = (\nu'_1, \dots, \nu'_k)$, a $k \times k$ matrix. Then,

$$|\det N| = |\det N_k| \nu_{k+1}^{k+1}.$$

Write $B'_i = B_1(o_i) \cap \{x_{k+1} = 0\}$ and regard it as a ball in $\mathbb{R}^k = \mathbb{R}^k \times \{0\}$. By induction, we have

$$\bigcap_{i=1}^k \partial B'_i = \{p, q'\},$$

and

$$|pq'| > \frac{|\det N_k|}{2^{k-2}}.$$

Without loss of generality, we assume $q' = (2a, 0, \dots, 0)$, with $a > |\det N_k|/2^{k-1}$. It is straightforward to verify

$$\bigcap_{i=1}^k \partial B_1(o_i) = \{(x_1, 0_{\mathbb{R}^{k-1}}, x_{k+1}) : (x_1 - a)^2 + x_{k+1}^2 = a^2\} := C_1.$$

Now we have

$$\begin{aligned} & \partial B_1(o_{k+1}) \cap \{(x_1, 0_{\mathbb{R}^{k-1}}, x_{k+1}) : x_1 \in \mathbb{R}, x_{k+1} \in \mathbb{R}\} \\ &= \{(x_1, 0_{\mathbb{R}^{k-1}}, x_{k+1}) : (x_1 - \nu_1^{k+1})^2 + (x_{k+1} - \nu_{k+1}^{k+1})^2 = (\nu_1^{k+1})^2 + (\nu_{k+1}^{k+1})^2\} := C_2. \end{aligned}$$

Note $0 \in C_1 \cap C_2$. By Lemma 6.1, there exists a point $q \in C_1 \cap C_2$ such that

$$|pq| = \frac{2aR \sin \alpha}{\sqrt{a^2 + R^2 - 2aR \cos \alpha}},$$

where $R = \sqrt{(\nu_1^{k+1})^2 + (\nu_{k+1}^{k+1})^2}$, $\cos \alpha = \nu_1^{k+1}/R$ and $\sin \alpha = \nu_{k+1}^{k+1}/R$. Therefore, with $a \leq 1$ and $R \leq 1$,

$$|pq| > a\nu_{k+1}^{k+1} > \frac{|\det N_k| \nu_{k+1}^{k+1}}{2^{k-1}} = \frac{|\det N|}{2^{k-1}}.$$

This implies the desired result for $n = k + 1$. \square

In fact, the proof above demonstrates that these spheres intersect at exactly two points.

In general for some $k \leq n$, let $\partial B_1(o_i)$ be k unit spheres in \mathbb{R}^n , $i = 1, \dots, k$, intersecting at a point p , such that the inner unit normal vectors ν_i of $\partial B_1(o_i)$ at p are linearly independent. Then, $\partial B_1(o_i)$ intersects at an $(n - k)$ -dimensional sphere, with its radius

$$r > \frac{\sqrt{\det(N^T N)}}{2^{k-2}},$$

where N is the matrix (ν_1, \dots, ν_k) . Here, the 0-dimensional sphere consists of two points. In particular, $\partial B_1(o_i)$ intersect each other at at least two points p and q , and

$$|pq| > \frac{\sqrt{\det(N^T N)}}{2^{k-2}}.$$

Without loss of generality, we assume $p = (a, 0, \dots, 0)$, and $q = (-a, 0, \dots, 0)$, where

$$a = \frac{|pq|}{2} > \frac{\sqrt{\det(N^T N)}}{2^{k-1}}.$$

Next, we set, for each $i = 1, \dots, k$,

$$\begin{aligned} \tilde{B}_i^1 &= \{x \in \mathbb{R}^n : x \in B_1^n(o_i)\}, \\ \tilde{B}_i^{-1} &= \{x \in \mathbb{R}^n : x \notin B_1^n(o_i)\}, \end{aligned}$$

and

$$(6.1) \quad \tilde{B}_{(l_1, \dots, l_k)} = \bigcap_{i=1}^k \tilde{B}_i^{l_i},$$

where $l_i = 1$ or -1 , for each $i = 1, \dots, k$. Then, we take

$$(6.2) \quad \tilde{B} = \bigcup_{(l_1, \dots, l_k)} \tilde{B}_{(l_1, \dots, l_k)}$$

where the union is over a fixed finite set of vectors (l_1, \dots, l_k) , with $l_i = 1$ or -1 for each $i = 1, \dots, k$. We require that \tilde{B} is a Lipschitz domain.

Similarly, we set, for each $i = 1, \dots, k$,

$$\begin{aligned} \hat{B}_i^1 &= \{x \in \mathbb{R}^n : x \notin \overline{B_1^n(o_i)}\}, \\ \hat{B}_i^{-1} &= \{x \in \mathbb{R}^n : x \in \overline{B_1^n(o_i)}\}, \end{aligned}$$

and

$$(6.3) \quad \hat{B}_{(l_1, \dots, l_k)} = \bigcap_{i=1}^k \hat{B}_i^{l_i},$$

where $l_i = 1$ or -1 , for each $i = 1, \dots, k$. Then, we take

$$(6.4) \quad \hat{B} = \bigcup_{(l_1, \dots, l_k)} \hat{B}_{(l_1, \dots, l_k)},$$

where the union is over the same set of vectors (l_1, \dots, l_k) as in (6.2).

In the following, we transform \tilde{B} and \hat{B} conformally to infinite cones. To this end, we embed \mathbb{R}^n into \mathbb{R}^{n+1} as $\mathbb{R}^n \times \{0\}$. Set

$$S^n(a) = \{(x_1, \dots, x_n, x_{n+1}) : x_1^2 + \dots + x_{n+1}^2 = a^2\}.$$

Consider the following three conformal transforms in \mathbb{R}^{n+1} . We always write $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Set

$$T_1(x, 0) = \left(\frac{2a^2 x_1}{a^2 + |x|^2}, \dots, \frac{2a^2 x_n}{a^2 + |x|^2}, \frac{a(a^2 - |x|^2)}{a^2 + |x|^2} \right).$$

Then, T_1 is the inverse transform of the stereographic projection which lifts $\mathbb{R}^n \times \{0\}$ to $S^n(a)$. Next, set

$$T_2(x_1, \dots, x_n, x_{n+1}) = (-x_{n+1}, x_2, \dots, x_n, x_1).$$

Then, T_2 is an orthogonal transform which rotates the $x_1 x_{n+1}$ -plane by $\pi/2$ is counter-clockwisely. Last, set

$$T_3(x, x_{n+1}) = \left(\frac{ax_1}{a + x_{n+1}}, \dots, \frac{ax_n}{a + x_{n+1}}, 0 \right).$$

Then, T_3 is the stereographic projection which transforms $S^n(a) \setminus (0_{\mathbb{R}^n}, -a)$ onto $\mathbb{R}^n \times \{0\}$. Next, we view $T_a = (T_3 T_2 T_1)|_{\mathbb{R}^n \times \{0\}}$ as a map in \mathbb{R}^n , which is given by

$$(6.5) \quad T_a x = \left(\frac{-a(a^2 - |x|^2)}{a^2 + 2ax_1 + |x|^2}, \frac{2a^2 x_2}{a^2 + 2ax_1 + |x|^2}, \dots, \frac{2a^2 x_n}{a^2 + 2ax_1 + |x|^2} \right).$$

Let V_p be the tangent cone of \tilde{B} at p . Therefore, T_a transforms \tilde{B} and \hat{B} conformally to infinite cones \tilde{V} and \hat{V} with vertices 0, which are conjugate to V_p . By a straightforward

calculation, the Jacobi matrix $(\frac{\partial T_a}{\partial x})$ has the form

$$(6.6) \quad \left(\frac{\partial T_a}{\partial x}(x) \right) = \frac{2a^2}{a^2 + 2ax_1 + |x|^2} O(x),$$

where $O(x)$ is an orthogonal matrix.

Let \tilde{v} be the solution of (1.1)-(1.2) for $\Omega = \tilde{V}$. Then, $\tilde{v}^{\frac{4}{n-2}}(y_1, \dots, y_n) \sum_{i=1}^n dy_i \otimes dy_i$ is a complete metric with the constant scalar curvature $-n(n-1)$ on \tilde{V} . Hence,

$$\tilde{u}^{\frac{4}{n-2}}(x) \sum_{i=1}^n dx_i \otimes dx_i = T_a^* (\tilde{v}^{\frac{4}{n-2}} \sum_{i=1}^n dy_i \otimes dy_i)$$

is a complete metric with the constant scalar curvature $-n(n-1)$ on \tilde{B} . A straightforward calculation, with the help of (6.6), yields

$$(6.7) \quad \tilde{u}(x) = \tilde{v}(T_a x) \left[\frac{2a^2}{a^2 + 2ax_1 + |x|^2} \right]^{\frac{n-2}{2}}.$$

Then, \tilde{u} is a solution of (1.1)-(1.2) for $\Omega = \tilde{B}$. In fact, by the scaling property of \tilde{v} and the explicit expression of T_a in (6.5), we have

$$\tilde{u}(x) = \tilde{v}(-a(a^2 - |x|^2), 2a^2 x_2, \dots, 2a^2 x_n) (2a^2)^{\frac{n-2}{2}}.$$

This expression will not be needed.

Similarly, let \hat{v} be the solution of (1.1)-(1.2) for $\Omega = \hat{V}$. Then,

$$\hat{u}(x) = \hat{v}(T_a x) \left[\frac{2a^2}{a^2 + 2ax_1 + |x|^2} \right]^{\frac{n-2}{2}},$$

and \tilde{v} is a solution of (1.1)-(1.2) for $\Omega = \tilde{B}$.

Now we are ready to prove the main theorem of this section.

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $x_0 \in \partial\Omega$ and, for some integer $k \leq n$, $\partial\Omega$ in a neighborhood of x_0 consists of k $C^{1,1}$ -hypersurfaces S_1, \dots, S_k intersecting at x_0 with the property that the normal vectors of S_1, \dots, S_k at x_0 are linearly independent. Suppose $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2) and $u_{V_{x_0}}$ is the corresponding solution in the tangent cone V_{x_0} . Then, for any x close to x_0 ,*

$$(6.8) \quad |u(x) - f_{V_{x_0}}(d_1, \dots, d_k)| \leq Cu(x)|x - x_0|,$$

where d_i is the signed distance to S_i with respect unit inner normal vectors, $f_{V_{x_0}}$ is the function $u_{V_{x_0}}$ in terms of d_1, d_2, \dots, d_k as in (4.5), C is a positive constant depending only on $n, \sigma(P_1, \dots, P_k)$ as defined in (4.6) and the $C^{1,1}$ -norms of S_1, \dots, S_k near x_0 .

Proof. Let ν_1, \dots, ν_k be the inner unit normal vectors of S_1, \dots, S_k at x_0 , respectively, and denote by N the matrix (ν_1, \dots, ν_k) as in (5.3). We fix an $x \in \Omega$ near $x_0 \in \partial\Omega$ and let p_i be the point on S_i with the least distance to x . Denote by ν_{p_i} the inner unit normal vector of S_i at p_i . Let $B_r(o_i)$ be the interior tangent ball of S_i at p_i with a radius r and $B_r(o'_i)$ be the exterior tangent ball of S_i at p_i , with r to be determined. We point

out that $B_r(o_i)$ is not necessarily in Ω and that $B_r(o'_i)$ is not necessarily outside Ω . We now divide the proof in several steps.

Step 1. We construct two sets, one inside Ω and one containing Ω . The set inside Ω is out of $B_r(o_1), \dots, B_r(o_k)$, while the set containing Ω is out of $B_r(o'_1)^c, \dots, B_r(o'_k)^c$.

We first prove that $\partial B_r(o_1), \dots, \partial B_r(o_k)$ intersect at at least two points with their distance bounded from below and that a similar result holds for $B_r(o'_i)$.

Denote by P_{p_i} the tangent plane of S_i at p_i . We assume $x = 0$, and

$$\nu_{p_i} = (\nu_1^{p_i}, \dots, \nu_k^{p_i}, 0_{\mathbb{R}^{n-k}}).$$

Consider the matrices

$$N_k = ((\nu_1^{p_1}, \dots, \nu_k^{p_1})^T, \dots, (\nu_1^{p_k}, \dots, \nu_k^{p_k})^T),$$

and

$$N' = (\nu_{p_1}, \dots, \nu_{p_k}).$$

Note that N_k is a $k \times k$ matrix and N' an $n \times k$ matrix. We can parameterize P_{p_i} as

$$\langle \nu_{p_i}, y - p_i \rangle = 0.$$

For $|x - x_0|$ small, we have

$$\|N_k^{-1}\|^2 = \|(N'^T N')^{-1}\| \leq \frac{\|(N'^T N')^{-1}\|}{1 - C\|(N'^T N')^{-1}\|} < 2\|(N'^T N')^{-1}\|,$$

where C is some positive constant depending only on n , $\|(N'^T N')^{-1}\|$, and the geometry of $\partial\Omega$. We have

$$|\langle \nu_{p_i}, p_i \rangle| = \langle \nu_{p_i}, p_i - x \rangle \leq |x - x_0|.$$

Consider the vector $b = (\langle \nu_{p_1}, p_1 \rangle, \dots, \langle \nu_{p_k}, p_k \rangle)^T \in \mathbb{R}^k$. Then the system of linear equations

$$\langle \nu_{p_i}, y - p_i \rangle = 0 \quad \text{for } i = 1, \dots, k,$$

has a solution $p = ((N_k^T)^{-1}b, 0_{\mathbb{R}^{n-k}})$ and $|p| \leq C|x - x_0|$.

We view the graph of $\partial B_r(o_i)$ and $\partial B_r(o'_i)$ near p as small perturbations of

$$\langle \nu_{p_i}, y - p_i \rangle = 0.$$

In other words, near 0, $\partial B_r(o_i)$ is parametrized by

$$\langle \nu_{p_i}, y - p_i \rangle = \tilde{g}_i(y),$$

and $\partial B_r(o'_i)$ is parametrized by

$$\langle \nu_{p_i}, y - p_i \rangle = \hat{g}_i(y),$$

where we have, for $y \in B_{C|x-x_0|}$,

$$|\tilde{g}_i(y)|, |\hat{g}_i(y)| \leq C_0(|x - x_0| + |y|)^2,$$

and

$$|D_y \tilde{g}_i(y)|, |D_y \hat{g}_i(y)| \leq C_0(|x - x_0| + |y|).$$

It is easy to verify that $G(y) = (N_k^T)^{-1}(\tilde{g}_i(y) + b)$ is a contraction mapping on $B_{C|x-x_0|} \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$, if $|x - x_0|$ is small. Therefore, the system of equations

$$y = G(y)$$

has a solution in $B_{C|x-x_0|} \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$. Hence, there exists a point \tilde{p} such that

$$\tilde{p} \in B_{C|x-x_0|} \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}) \cap \left(\bigcap_{i=1}^k \partial B_r(o_i) \right).$$

Denote by $\tilde{\nu}_i$ the inner normal vector of $B_r(o_i)$ at \tilde{p} , and by \tilde{N} the matrix $(\tilde{\nu}_1, \dots, \tilde{\nu}_k)$. For $|x - x_0|$ small, we have

$$|\det(\tilde{N}^T \tilde{N})| = |\det N^T N| |\det[I + (N^T N)^{-1}(\tilde{N}^T \tilde{N} - N^T N)]| \geq \frac{1}{4} |\det N^T N|,$$

and

$$\|(\tilde{N}^T \tilde{N})^{-1}\| \leq \frac{\|(N^T N)^{-1}\|}{1 - C\|(N^T N)^{-1}|x - x_0|\|} < 2\|(N^T N)^{-1}\|.$$

By Lemma 6.2, we have

$$\bigcap_{i=1}^k \partial B_r(o_i) \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}) = \{\tilde{p}, \tilde{q}\},$$

and

$$|\tilde{p}\tilde{q}| > r \frac{|\sqrt{\det \tilde{N}^T \tilde{N}}|}{2^{k-2}} > r \frac{\sqrt{\det N^T N}}{2^{k-1}}.$$

Similarly, there exists a point \hat{p} ,

$$\hat{p} \in B_{C|x-x_0|} \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}) \cap \left(\bigcap_{i=1}^k \partial B_r(o'_i) \right).$$

Denote by $\hat{\nu}_i$ the unit outer normal vector of $B_r(o'_i)$ at \hat{p} , and by \hat{N} the matrix $(\hat{\nu}_1, \dots, \hat{\nu}_k)$. For $|x - x_0|$ small, we have

$$|\det \hat{N}^T \hat{N}| = |\det N^T N| |\det[I + (N^T N)^{-1}(\hat{N}^T \hat{N} - N^T N)]| \geq \frac{1}{4} |\det N^T N|,$$

and

$$\|(\hat{N}^T \hat{N})^{-1}\| \leq \frac{\|(N^T N)^{-1}\|}{1 - C\|(N^T N)^{-1}|x - x_0|\|} < 2\|(N^T N)^{-1}\|.$$

Moreover,

$$\bigcap_{i=1}^k \partial B_r(o'_i) \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}) = \{\hat{p}, \hat{q}\},$$

and

$$|\hat{p}\hat{q}| > 2r \frac{\sqrt{\det \hat{N}^T \hat{N}}}{2^{k-1}} > r \frac{\sqrt{\det N^T N}}{2^{k-1}}.$$

We now construct two sets, one inside Ω and another containing Ω .

Suppose, for some constant $R > 0$, $\Omega \cap B_R(x_0)$ can be expressed by the union of some $\Omega_{(l_1, \dots, l_k)}$, i.e.,

$$(6.9) \quad \Omega = \bigcup \Omega_{(l_1, \dots, l_k)},$$

where the union is over a finite set of vectors (l_1, \dots, l_k) , with $l_i = 1$ or -1 for each $i = 1, \dots, k$. Refer to discussions in Section 4.

With $B_r(o_i)$ replacing $B_1^n(o_i)$, we can define $\tilde{B}_{(l_1, \dots, l_k)}$ as in (6.1), for any (l_1, \dots, l_k) , with $l_i = 1$ or -1 for each $i = 1, \dots, k$. Then, we set

$$(6.10) \quad \tilde{B} = \bigcup \tilde{B}_{(l_1, \dots, l_k)},$$

where the union is over the same set of vectors (l_1, \dots, l_k) as in (6.9). We note that \tilde{B} is a Lipschitz domain.

Similarly, with $B_r(o'_i)$ replacing $B_1^n(o'_i)$, we can define $\hat{B}_{(l_1, \dots, l_k)}$ as in (6.3), for any (l_1, \dots, l_k) , with $l_i = 1$ or -1 for each $i = 1, \dots, k$. Then, we set

$$(6.11) \quad \hat{B} = \bigcup \hat{B}_{(l_1, \dots, l_k)},$$

where the union is over the same set of vectors (l_1, \dots, l_k) as in (6.9). Similarly, \hat{B} is a Lipschitz domain.

For some small constants r and r^* depending only on the geometry of $\partial\Omega$, we have $\tilde{B} \subseteq \Omega$ and $\Omega \subseteq \tilde{B}$ if $|x - x_0| \leq r^*$. We can check this by Lemma 4.6(ii). We point out that r^* is relatively small compared with r . We also note that each ball $B_r(o_i)$ is above the corresponding hypersurface S_i , although it is not necessarily in Ω , and that each ball $B_r(o'_i)$ is below the corresponding hypersurface S_i .

Step 2. We now compare the solution u with the corresponding solution in the tangent cone.

Let \tilde{u} be the solution of (1.1)-(1.2) for $\Omega = \tilde{B}$. By the maximum principle, we have $u \leq \tilde{u}$ in \tilde{B} . In particular, we have $u(x) \leq \tilde{u}(x)$. By a translation and a rotation, we assume $\tilde{p} = (\tilde{a}, 0, \dots, 0)$, and $\tilde{q} = (-\tilde{a}, 0, \dots, 0)$, where

$$(6.12) \quad \tilde{a} = \frac{|\tilde{p}\tilde{q}|}{2} > r \frac{|\sqrt{\det N^T N}|}{2^k}.$$

Let $T_{\tilde{a}}$ be the conformal transform given by (6.5), with \tilde{a} replacing a , and \tilde{v} be the solution of (1.1)-(1.2) for $\Omega = \tilde{V} = T_{\tilde{a}}(\tilde{B})$. Then, by (6.6) and (6.7), the Jacobi matrix $(\frac{\partial T_{\tilde{a}}}{\partial x})$ has the form

$$\left(\frac{\partial T_{\tilde{a}}}{\partial x}(x) \right) = \frac{2\tilde{a}^2}{\tilde{a}^2 + 2\tilde{a}x_1 + |x|^2} O(x),$$

where $O(x)$ is an orthogonal matrix, and

$$\tilde{u}(x) = \tilde{v}(T_{\tilde{a}}x) \left(\frac{2\tilde{a}^2}{\tilde{a}^2 + 2\tilde{a}x_1 + |x|^2} \right)^{\frac{n-2}{2}}.$$

Let \tilde{P}_i be the hyperplane transformed from $\partial B_r(o_i)$ by the map $T_{\tilde{a}}$. Denote by \tilde{d}_i the signed distance from $T_{\tilde{a}}x$ to \tilde{P}_i . We have

$$|x - \tilde{p}| \leq C|x - x_0|.$$

By the explicit form of the Jacobi matrix $(\frac{\partial T_{\tilde{a}}}{\partial x})$, we have, for $|x - x_0|$ small,

$$\tilde{d}_i = \left[\frac{2\tilde{a}^2}{\tilde{a}^2 + 2\tilde{a}x_1 + |x|^2} + O(|x - x_0|) \right] d_i.$$

Then, by using $\tilde{x} = (\tilde{a}, 0, \dots, 0)$ at \tilde{p} and the lower bound of \tilde{a} in (6.12), we obtain

$$\tilde{d}_i = \left(\frac{1}{2} + O(|x - x_0|) \right) d_i.$$

We now write $\tilde{v} = f_{\tilde{V}}(\tilde{d}_1, \dots, \tilde{d}_k)$ as in (4.5). By the scaling property (2.3) and Lemma 3.5, we have

$$f_{\tilde{V}}(\tilde{d}_1, \dots, \tilde{d}_k) \leq f_{\tilde{V}}(d_1, \dots, d_k) \left(\frac{1}{2} - C|x - x_0| \right)^{-\frac{n-2}{2}}.$$

Therefore,

$$u(x) \leq \tilde{u}(x) = f_{\tilde{V}}(\tilde{d}_1, \dots, \tilde{d}_k) \left[\frac{2\tilde{a}^2}{\tilde{a}^2 + 2\tilde{a}x_1 + |x|^2} \right]^{\frac{n-2}{2}} \leq f_{\tilde{V}}(d_1, \dots, d_k)(1 + C|x - x_0|).$$

Note $\|\tilde{N} - N\| \leq C|x - x_0|$. By Lemma 5.3, we have

$$f_{\tilde{V}}(d_1, \dots, d_k) \leq f_{V_{x_0}}(d_1, \dots, d_k)(1 + C|x - x_0|).$$

Therefore,

$$(6.13) \quad u(x) \leq f_{V_{x_0}}(d_1, \dots, d_k)(1 + C|x - x_0|).$$

Let \hat{u} be the solution of (1.1)-(1.2) for $\Omega = \hat{B}$. By the maximum principle, $\hat{u} \leq u$ in Ω . In particular, we have $\hat{u}(x) \leq u(x)$. By a translation and a rotation, we assume $\hat{p} = (\hat{a}, 0, \dots, 0)$ and $\hat{q} = (-\hat{a}, 0, \dots, 0)$, where

$$\hat{a} = \frac{|\hat{p}\hat{q}|}{2} > r \frac{|\sqrt{\det N^T N}|}{2^k}.$$

Similarly, we get

$$u(x) \geq \hat{u}(x) = \hat{v}(T_{\hat{a}}x) \left[\frac{2\hat{a}^2}{\hat{a}^2 + 2\hat{a}x_1 + |x|^2} \right]^{\frac{n-2}{2}} \geq f_{\hat{V}}(d_1, \dots, d_k)(1 - C|x - x_0|),$$

where $T_{\hat{a}}$ is the conformal transform given by (6.5), with \hat{a} replacing a , \hat{v} is the solution of (1.1)-(1.2) for $\Omega = \hat{V} = T_{\hat{a}}(\hat{B})$ and $\hat{v} = f_{\hat{V}}(\hat{d}_1, \dots, \hat{d}_k)$ as in (4.5). By Lemma 5.3, we have

$$f_{\hat{V}}(d_1, \dots, d_k) \geq f_{V_{x_0}}(d_1, \dots, d_k)(1 - C|x - x_0|).$$

Therefore,

$$(6.14) \quad u(x) \geq f_{V_{x_0}}(d_1, \dots, d_k)(1 - C|x - x_0|).$$

Combining (6.13) and (6.14), we complete the proof. \square

Now we compare the proof of Theorem 6.3 and that of Theorem 3.1. In the proof of Theorem 3.1, we construct two balls, one inside Ω and one outside Ω , and then compare the solution u with the corresponding solution in the interior ball and the solution outside the exterior ball. In the proof of Theorem 6.3, we replace the interior ball and the complement of the exterior ball by \tilde{B} and \hat{B} , constructed from the interior tangent balls and the complements of the exterior tangent balls, respectively, and then compare the solution u with the corresponding solutions in these two sets. The conformal structure of the equation plays an essential role in the present proof.

We are ready to prove the main result in this paper.

Proof of Theorem 1.1. We adopt the notations from Theorem 6.3 and its proof. Let Ω be bounded by $C^{1,1}$ hypersurfaces S_1, \dots, S_k near $x_0 \in \partial\Omega$ and the tangent cone V_{x_0} of Ω at x_0 be bounded by P_1, \dots, P_k , the tangent planes of S_1, \dots, S_k at x_0 , respectively, with ν_1, \dots, ν_k the inner unit normal vectors.

First, we consider the case $k = n$. Without loss of generality, we assume x_0 is the origin. For any x sufficiently small, we define

$$T_{\{S_i\}}x = (d_1(x), \dots, d_n(x)),$$

where $d_i(x)$ is the signed distance from x to S_i with respect to ν_i , $i = 1, \dots, n$. We emphasize that $T_{\{S_i\}}$ is defined in a full neighborhood of the origin instead of only in Ω and that the signed distance is used instead of its absolute value. Then, $T_{\{S_i\}}$ is $C^{1,1}$ near the origin and its Jacobi matrix at the origin is nonsingular by the linear independence of ν_1, \dots, ν_n . Therefore, $T_{\{S_i\}}$ is a $C^{1,1}$ -diffeomorphism in a neighborhood of the origin. We have a similar result for $T_{\{P_i\}}$, with P_1, \dots, P_n replacing S_1, \dots, S_n . In fact, $T_{\{P_i\}}$ is a linear transform, since P_1, \dots, P_n are hyperplanes passing the origin. Then, the map $T = T_{\{P_i\}}^{-1} \circ T_{\{S_i\}}$ is a $C^{1,1}$ -diffeomorphism near the origin and has the property that the signed distance from x to S_i is the same as that from Tx to P_i , for $i = 1, \dots, n$.

Next, we consider the case $k < n$. We add hyperplanes P_{k+1}, \dots, P_n passing the point x_0 with their unit normal vectors ν_{k+1}, \dots, ν_n forming an orthonormal basis of the orthogonal complement of $\text{Span}\{\nu_1, \dots, \nu_k\}$, as in the proof of Theorem 5.3. We denote by d_j the signed distance from x to P_j with respect to ν_j , $j = k+1, \dots, n$. Then we can construct the map T as in the case $k = n$. \square

We point out that the assumption $k \leq n$ plays an essential role in the proof of the optimal estimate (1.5) as stated in Theorem 1.1. For example, $k = n$ is needed crucially in the identification of the points by their signed distances to n $C^{1,1}$ -hyperplanes as in Section 4 and hence in the construction of the transform T . Moreover, the assumption $k \leq n$ is used to find an intersect of k spheres near x_0 as in Step 1 of the proof of Theorem 6.3 and to obtain a lower bound of the distance between this intersect and another intersect away from x_0 as in Lemma 6.2.

7. SOLUTIONS IN GENERAL SINGULAR DOMAINS

In this section, we study asymptotic expansions near singular points for more general domains. Specifically, we allow $k > n$ in Theorem 6.3 and derive optimal estimates for points strictly located inside the tangent cone. The discussion again relies essentially on the conformal invariance of the equation (1.1).

Let Ω be a bounded Lipschitz domain. We fix a point $x_0 \in \partial\Omega$ and assume that it is the origin such that, for some $R > 0$,

$$\Omega \cap B_{2R}(x_0) = \{x \in B_{2R}(x_0) : x_n > f(x')\},$$

for some Lipschitz function f on B'_{2R} with $f(0) = 0$. Then, there exists a finite circular cone V_{θ_0} , with x_0 as its vertex, x_n -axis as the axis of the cone, an apex angle $2\theta_0$ and a height h , such that

$$(7.1) \quad V_{\theta_0} \subseteq \overline{\Omega}, \quad -V_{\theta_0} \subseteq \Omega^c,$$

where $-V_{\theta_0}$ is the reflection of V_{θ_0} about $x_n = 0$ and θ_0 and h are constants depending only on the geometry of $\partial\Omega$.

Fix an integer $k \geq 2$. Recall the domain Ω introduced in Definition 4.5 and the subsequent discussion of its decomposition and its tangent cones.

We now prove an important property concerning tangent cones.

Lemma 7.1. *For some point $x_0 \in \partial\Omega$, let Ω be a bounded Lipschitz domain bounded by k $C^{1,1}$ -hypersurfaces S_1, \dots, S_k near x_0 as in Definition 4.5 and let V_{x_0} be the tangent cone of Ω at x_0 in the setting following Definition 4.5. Then, for any small r ,*

$$\left(V_{x_0} + \frac{Mr^2}{\sin \theta_0} e_n\right) \cap B_r(x_0) \subset \Omega,$$

and

$$\Omega \cap B_r(x_0) \subset V_{x_0} - \frac{Mr^2}{\sin \theta_0} e_n,$$

where M is the maximum of the $C^{1,1}$ -norms of S_1, \dots, S_k near x_0 and θ_0 is introduced for (7.1).

Proof. We assume x_0 is the origin. Let P'_i and P''_i be the hyperplanes transformed from P_i in the direction of ν_i and $-\nu_i$ by a distance of r^2 , respectively. Assume that the hyperplane P_i is expressed by $x_n = L_i(x')$ for some linear function L_i and that the hypersurface S_i near x_0 is expressed by $x_n = f_i(x')$ for some $C^{1,1}$ -function f_i , for each $i = 1, 2, \dots, k$. Note, by adjusting M ,

$$L_i(x') - M|x'|^2 \leq f_i(x') \leq L_i(x') + M|x'|^2.$$

By

$$M \left((M^{-1}r)^{\frac{1}{2}} \right)^2 = r,$$

we have

$$(7.2) \quad \left(V_{x_0} + \frac{r}{\sin \theta_0} e_n\right) \cap B_{(M^{-1}r)^{\frac{1}{2}}}(x_0) \subset \Omega,$$

and

$$(7.3) \quad \left(\Omega \cap B_{(M^{-1}r)^{\frac{1}{2}}}(x_0) \right) \subset V_{x_0} - \frac{r}{\sin \theta_0} e_n.$$

In fact, take any $x \in \left(V_{x_0} + \frac{r}{\sin \theta_0} e_n \right) \cap B_{(M^{-1}r)^{\frac{1}{2}}}(x_0)$. Then, for some (l_1, \dots, l_k) in (6.9),

$$x \in \left(V_{(l_1, \dots, l_k)} + \frac{r}{\sin \theta_0} e_n \right) \cap B_{(M^{-1}r)^{\frac{1}{2}}}(x_0),$$

and, as a consequence, for some $(l'_1, \dots, l'_k) \geq (l_1, \dots, l_k)$,

$$x \in \left(\Omega_{(l'_1, \dots, l'_k)} + \frac{r}{\sin \theta_0} e_n \right) \cap B_{(M^{-1}r)^{\frac{1}{2}}}(x_0),$$

since the graph of $P_i + \frac{r}{\sin \theta_0} e_n$ is above the graph of S_i in $B'_{(M^{-1}r)^{\frac{1}{2}}}(x_0)$. Hence, $x \in \Omega$. This proves (7.2). A similar argument yields (7.3). We have the desired result by renaming radii in (7.2) and (7.3). \square

Lemma 7.1 asserts the following statement: In a neighborhood of x_0 , if V_{x_0} is translated in the direction of e_n by an appropriate distance, its graph is above the graph of $\partial\Omega$, and if V_{x_0} is translated in the direction of $-e_n$ by the same distance, its graph is below the graph of $\partial\Omega$.

We now prove the main result in this section.

Theorem 7.2. *For some point $x_0 \in \partial\Omega$, let Ω be a bounded Lipschitz domain bounded by k $C^{1,1}$ -hypersurfaces S_1, \dots, S_k near x_0 as in Definition 4.5 and let V_{x_0} be the tangent cone of Ω at x_0 . Suppose that $u \in C^\infty(\Omega)$ is a solution of (1.1)-(1.2) and that v is the corresponding solution in V_{x_0} . Then, for any $\delta > 0$ and any $x \in \Omega$ close to x_0 with $\text{dist}(x, \partial\Omega) > \delta|x - x_0|$,*

$$(7.4) \quad |u(x) - v(x)| \leq C\delta^{-1}u(x)|x - x_0|,$$

where C is a positive constant depending only on n , R , θ_0 and the $C^{1,1}$ -norms of hypersurfaces S_1, \dots, S_k near x_0 .

Proof. We fix an $x \in \Omega$ near $x_0 \in \partial\Omega$ with $\text{dist}(x, \partial\Omega) > \delta|x - x_0|$. We denote by ν_i the interior unit normal vector of S_i at x_0 . Then, by (7.1),

$$(7.5) \quad \langle \nu_i, e_n \rangle > \sin \theta_0.$$

For some constant $r > 0$, set

$$r_i = \frac{r}{\langle \nu_i, e_n \rangle},$$

and

$$\begin{aligned} \tilde{B}_i &= B_{r_i}(x_0 + r_i \nu_i), \\ \hat{B}_i &= B_{r_i}(x_0 - r_i \nu_i). \end{aligned}$$

By (7.5), all r_i are comparable. Then, we have

$$x_0, x_0 + 2re_n \in \bigcap_{i=1}^k \partial \tilde{B}_i,$$

and

$$x_0, x_0 - 2re_n \in \bigcap_{i=1}^k \partial \hat{B}_i.$$

For some constant r depending only on R and the $C^{1,1}$ -norm of S_i , we note that each ball \tilde{B}_i is above the corresponding hypersurface S_i , although it is not necessarily in Ω , and that each ball \hat{B}_i is below the corresponding hypersurface S_i .

For some constant $R > 0$, $\Omega \cap B_R(x_0)$ can be expressed by the union of some $\Omega_{(l_1, \dots, l_k)}$, i.e.,

$$(7.6) \quad \Omega = \bigcup \Omega_{(l_1, \dots, l_k)},$$

where the union is over a finite set of vectors (l_1, \dots, l_k) , with $l_i = 1$ or -1 for each $i = 1, \dots, k$.

With \tilde{B}_i replacing $B_1^n(o_i)$, we can define $\tilde{B}_{(l_1, \dots, l_k)}$ as in (6.1), for any (l_1, \dots, l_k) with $l_i = 1$ or -1 for each $i = 1, \dots, k$. Then, we set

$$(7.7) \quad \tilde{B} = \bigcup \tilde{B}_{(l_1, \dots, l_k)},$$

where the union is over the same set of vectors (l_1, \dots, l_k) as in (7.6). We note that \tilde{B} is a Lipschitz domain.

Similarly, with \hat{B}_i replacing $B_1^n(o'_i)$, we can define $\hat{B}_{(l_1, \dots, l_k)}$ as in (6.3), for any (l_1, \dots, l_k) with $l_i = 1$ or -1 for each $i = 1, \dots, k$. Then, we set

$$(7.8) \quad \hat{B} = \bigcup \hat{B}_{(l_1, \dots, l_k)},$$

where the union is over the same set of vectors (l_1, \dots, l_k) as in (7.6). Similarly, \hat{B} is a Lipschitz domain.

For some small constants r and r^* depending only on the geometry of $\partial\Omega$, we have $\tilde{B} \subseteq \Omega$ and $\Omega \subseteq \hat{B}$ if $|x - x_0| \leq r^*$. By Lemma 7.1, we have, for $x \in \tilde{B}$ with $|x - x_0|$ small,

$$\text{dist}(x, \partial \tilde{B}) \geq \text{dist}(x, \partial \Omega) - C|x - x_0|^2,$$

and

$$\text{dist}(x, \partial \hat{B}) \leq \text{dist}(x, \partial \Omega) + C|x - x_0|^2.$$

Let \tilde{u} be the solution of (1.1)-(1.2) for $\Omega = \tilde{B}$. By the maximum principle, we have $u \leq \tilde{u}$ in \tilde{B} . In particular, we have $u(x) \leq \tilde{u}(x)$.

By a translation and a rotation, we assume $x_0 = (r, 0, \dots, 0)$, and $x_0 + 2re_n = (-r, 0, \dots, 0)$. Let T_r be the conformal transform given by (6.5), with r replacing a , and

let \tilde{v} be the solution of (1.1)-(1.2) for $\Omega = \tilde{V} = T_r(\tilde{B})$. Then, $T_r(\tilde{B}) = V_{x_0} - re_1$ and

$$\tilde{u}(x) = \tilde{v}(T_r x) \left(\frac{2r^2}{r^2 + 2rx_1 + |x|^2} \right)^{\frac{n-2}{2}}.$$

Note

$$\left(\frac{\partial T_r}{\partial x}(x_0) \right) = \frac{1}{2} I_{n \times n}.$$

Hence,

$$|(T_r x - T_r x_0) - \frac{1}{2}(x - x_0)| = O(|x - x_0|^2).$$

Note, for $|x - x_0|$ small,

$$\text{dist}(x, \partial V_{x_0}) > \frac{\delta}{2} |x - x_0|.$$

Then, by Lemma 3.4, we have

$$|\tilde{v}(T_r x) - 2^{\frac{n-2}{2}} v(x)| \leq C \delta^{-1} v(x) |x - x_0|.$$

Therefore,

$$(7.9) \quad u(x) \leq \tilde{u}(x) = \tilde{v}(T_r x) \left[\frac{2r^2}{r^2 + 2rx_1 + |x|^2} \right]^{\frac{n-2}{2}} \leq v(x)(1 + C \delta^{-1} |x - x_0|).$$

Similarly, we can prove

$$(7.10) \quad u(x) \geq v(x)(1 - C \delta^{-1} |x - x_0|).$$

Combining (7.9) and (7.10), we have the desired result. \square

We note that the constant C in Theorem 7.2 depends on the $C^{1,1}$ -norms of S_1, \dots, S_k but independent of the number of hypersurfaces. Appropriately modified, Theorem 7.2 allows us to discuss asymptotic expansions near singular points of other types. For example, if V_{x_0} has an isolated singularity (at its vertex), we can approximate V_{x_0} by a sequence of cones V_k such that the number of faces of V_k approaches the infinity as $k \rightarrow \infty$ and the $C^{1,1}$ -norms of ∂V_k remain uniformly bounded. As a consequence, a result similar as Theorem 7.2 holds for this class of domains.

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